Chapter -1-

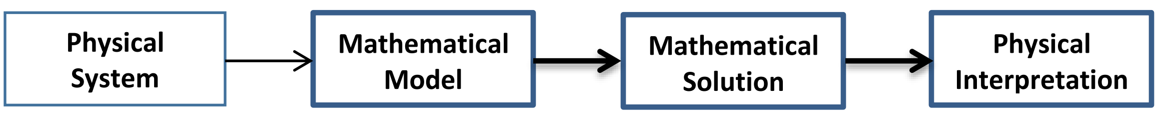
Introduction to Differential Equations

1.1 Introduction:

Definition of model: A model is a simplified representation of a system at some particular point in time and/or space intended to promote understanding of the real system.

Definition of system: A system typically consists of components (or elements) which are connected together in order to facilitate the flow of information, matter or energy…

Definition of Modeling: is the process whereby a physical situation or some other observation is translated into a mathematical model in terms of variables, functions, equations. The following Figure shows the modeling steps



1.1.1 Model Representation

The following groups of variables are commonly used to describe a chemical process: input variables, state variables, and output variables.

In addition to the above three groups of variables, parameters or constant parameters are also present in the mathematical model of the chemical processes. A parameter is typically a physical or chemical property value. The values of parameters, such as density, heat transfer coefficient, mass transfer coefficient, etc., must be known or specified for solving a problem.

All the process variables depend on time and/or spatial position. Therefore, they all are considered as *dependent* variables. Whereas, time and the spatial coordinate variables are the *independent* variables.

1.1.2 Types of Modelling Equations

A mathematical process model usually consists of the following types of equations:

- Algebraic equations (AEs)

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

- Ordinary differential equations (ODEs)

$$\frac{dy}{dx} = f[y(t)]$$

- Partial differential equations (PDEs) used with unknown function of two or more variables such as function *u* of two variables (*x*, *y*)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

1.1.3 Differential Equation

An equation containing the derivatives of one or more unknown functions (or dependent variables), with respect to one or more independent variables, is said to be a **differential equation (DE).**

➢ Classification by Type

- **Ordinary differential equation (ODE):** If a differential equation contains only ordinary derivatives of one or more unknown functions with respect to a *single* independent variable,

$$\frac{dy}{dx} + 5y = e^x$$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0$$

$$\frac{dx}{dt} + \frac{dy}{dt} = 2x + y \ (an\ ODE\ with\ two\ unkown\ functions)$$

- **Partial differential equation (PDE).** An equation involving partial derivatives of one or more unknown functions of two or more independent variables

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}-2\frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

➢ Notation

Notation the ordinary derivatives will be written by using either the Leibniz notation or prime notation

- Leibniz notation

$$\frac{dy}{dx},\frac{d^2y}{dx^2},\frac{d^ny}{dx^n}$$

The advantage of Leibniz notation is that it clearly display both dependent and independent variable

$$\frac{d^2x}{dt^2}+16x=0$$

where x is the unknown function or dependent variable and t is the independent variable

- Prime notation

$$y',y'',y'''$$

The differential equation can be written more compactly. The prime notation is used to denote only the first three derivatives. The 4th derivative of y is written $y^{(4)}$. The nth derivative of y is written $y^{(n)}$.

Partial derivatives are often denoted by a **subscript notation** indicating the independent variables. For example

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}-2\frac{\partial u}{\partial t}\ becomes\ u_{xx}=u_{tt}-u_t$$

➢ Classification by Order

The **order of a differential equation** (either ODE or PDE) is the order of the highest derivative in the equation.

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x \text{ is second order}$$

➢ Ordinary Differential Equation Form

An n^{th} order ODE in one dependent variable can be expressed by the general form

$$F\left(x, y, y', y'', \dots \dots., y^{(n)}\right) = 0$$

where *F* is a real valued function

The normal form of ordinary differential equation is

$$\frac{d^ny}{dx^n} = f(x, y, y', y'', \dots \dots., y^{(n-1)})$$

where *f* is a real valued continuous function.

Example: Represent the following ODEs as the normal form

(1) $4xy' + y = x \implies y' = (x - y)/4x$

(2) $y'' - y' + 6y = 0 \implies y'' = y' - 6y$

➢ Classification of linearity

An nth-order ordinary differential equation is said to be linear if *F* is linear in $y, y', y'', \dots \dots., y^{(n)}$

this means that an n^{th} order ODE is linear when

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)}y' + \dots + a_1(x)y' + a_0(x)y - g(x) = 0$$

The two properties of a linear ODE are as follows

- The dependent variable *y* and all its derivatives are the first degree (the power of each term is 1)
- The coefficients $a_0, a_1, \dots, a_n$ depend at most on the independent variable *x*

For example, $y'' - 2y' + y = 0, x^3\frac{d^3y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$

A **nonlinear** ordinary differential equation is simply one that is not linear. Nonlinear functions of the dependent variable or its derivatives, such as sin y or $e^{y'}$, cannot appear in a linear equation. For example, $\frac{d^4y}{dx^4} + y^2 = 0, \frac{d^2y}{dx^2} + \sin y = 0, (1-y)\, y' + 2y = e^x$

Example: Check the following ODE is linear or nonlinear

$$(y-x)dx + 4xdy = 0 \implies 4xy' + y = x \; is \; linear$$

1.1.4 Solution of ODE

A solution of an nth order ODE is any function on an interval I that processes at least n derivatives that are continuous on the interval I which is the domain of the solution. When a solution substituted into nth order ODE satisfies the differential equation on I.

For example, the differential equation $\frac{dy}{dx} = xy^{\frac{1}{2}}$ has the solution $y = \frac{1}{16}x^4$ and the constant solution y =0, $-\infty < x < \infty$.

Trivial solution is the solution y =0

Example: Verify that the indicated function is a solution of the given differential equation on the interval $(-\infty, \infty)$

$$(1)\frac{dy}{dx} = xy^{\frac{1}{2}}\,;\; y = \frac{1}{16}x^4$$

$$(2)\quad y'' - 2y' + y = 0\,; y = xe^x$$

(1) Left hand side:

$$\frac{dy}{dx} = \frac{1}{16}(4.x^3) = \frac{1}{4}x^3$$

Right hand side:

$$xy^{\frac{1}{2}} = x.\left(\frac{1}{16}x^4\right)^{1/2} = x.\left(\frac{1}{4}x^2\right) = \frac{1}{4}x^3$$

(2) Left hand side:

$$y'' - 2y' + y = (xe^x + 2e^x) - 2(xe^x + 2e^x) + xe^x = 0$$

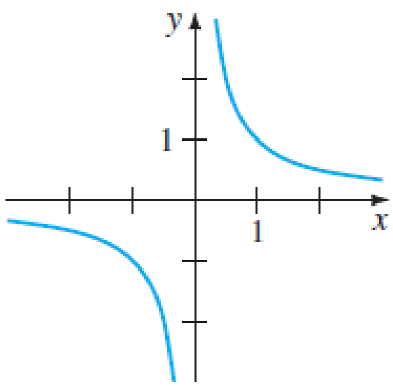
Right hand side: 0

➢ Solution Curve

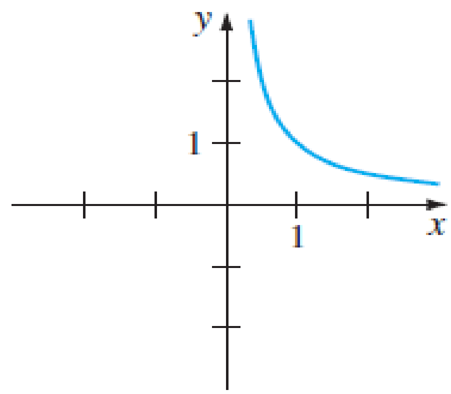
The graph of a solution $\emptyset$ of an ODE is called a **solution curve.** Since $\emptyset$ is a differentiable function, it is continuous on its interval *I* of definition.

For example, the differential equation $xy' + y = 0$ has the solution $y = 1/x$ the domain of the function is the set of all real numbers *x* except 0. This function is not differentiable at x =0.

Now $y = 1/x$ is also a solution of the linear first-order differential equation. Thus, $y = 1/x$ is a solution of the DE on *any* interval that does not contain 0.



(a) function $y = 1/x$, $x \neq 0$



(b) solution $y = 1/x$, $(0, \infty)$

➢ Explicit and Implicit Solutions

A solution in which the dependent variable is expressed solely in terms of the independent variable and constants as an explicit formula $y = \emptyset(x)$ is said to be an **explicit solution.**

For example, $y = \frac{1}{16}x^4, y = xe^x, and\ y = \frac{1}{x}$ are, in turn, explicit solutions of $\frac{dy}{dx} = xy^{1/2}, y'' - 2y' + y = 0, and\ xy' + y = 0.$ Moreover, the trivial solution *y* =0 is an explicit solution of all three equations.

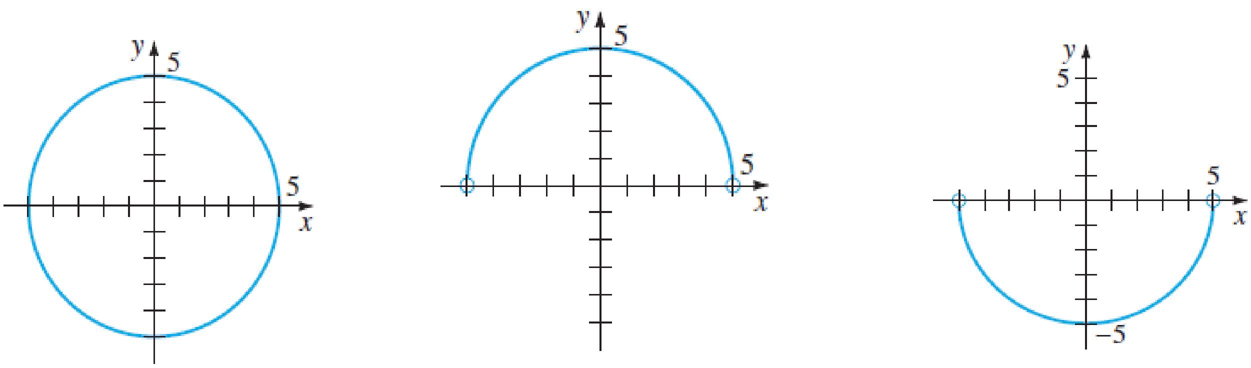
A relation *G*(*x*, *y*) =0 is said to be an **implicit solution** of an ordinary differential equation on an interval *I*, provided that there exists at least one function $\emptyset$ that satisfies the relation as well as the differential equation on *I*.

For example,

The relation $x^2 + y^2 = 25$on the open interval (-5, 5) is an implicit solution of the differential equation

$$\frac{dy}{dx} = -\frac{x}{y}$$

Solving $x^2 + y^2 = 25$ for y in terms of x yields $y = \pm\sqrt{25 - x^2}$. The two functions $y = \emptyset_1(x) = +\sqrt{25 - x^2}$ *and* $y = \emptyset_2(x) = -\sqrt{25 - x^2}$ satisfy the relation and are explicit solutions defined on the interval (-5, 5).



(a) implicit solution $x^2 + y^2 = 25$

(b) explicit solution $y_1 = \sqrt{25 - x^2}, -5 < x < 5$

(c) explicit solution $y_2 = -\sqrt{25 - x^2}, -5 < x < 5$

➢ Families of Solutions

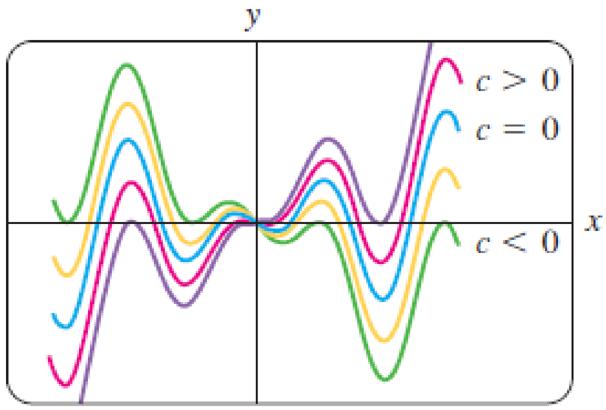
When solving a first-order differential equation *F(x, y, y′)=0*, the solution usually contains a single arbitrary constant or parameter c. A solution containing an arbitrary constant represents a set *G(x, y, c) =0* of solutions called a **one-parameter family of solutions.**

When solving an nth-order differential equation $F(x, y, y', \dots, y^{(n)}) = 0$, an **n-parameter family of solutions** $G(x, y, c_1, c_2, \dots, c_n) = 0$ are obtained.

This means that a single differential equation can have an infinite number of solution corresponding to the unlimited number of choices for the parameter(s). The family of solutions is the **general solution** of the differential equation. A solution of a differential equation that is free of arbitrary parameters is called a **particular solution.**

For example,

The one-parameter family $y = cx - x\cos x$ is an explicit solution of the linear first-order equation $xy' - y = x^2 \sin x$ on the interval $(-\infty, \infty)$. The following Figure shows the graphs of some particular solutions in this family for various choices of c.



Sometimes a differential equation possesses a solution that is not a member of a family of solutions of the equation that is, a solution that cannot be obtained by specializing *any* of the parameters in the family of solutions. Such an extra solution is called a **singular solution.**

For example, $y = \left(\frac{1}{4}x^2 + \mathrm{c}\right)^2$ *and* y =0 are solutions of the differential equation $\frac{dy}{dx} = xy^{1/2}$ *on* $(-\infty, \infty)$. the trivial solution *y* =0 is a singular solution, since it is not a member of the family ; there is no way of assigning a value to the constant *c* to obtain *y* =0.

Exercises:

(1) Identify the unknown function, independent variable, type and order of each differential equation:

1. $\frac{\partial^2}{\partial x^2}u + 3\left(\frac{\partial^2}{\partial y \partial x}u\right) = \frac{\partial^2}{\partial y^2}u$
2. $y''' + 5xy' + y = e^x + 2$
3. $y'' - e^{3xy} = \tan(x)$
4. $y' = 3x + \ln(x)$
5. $\frac{d^2y}{dx^2} - 4y\left(\frac{dy}{dx}\right) + x^2y = \frac{d^3y}{dx^3}$

6. $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}-e^{xy}=\frac{\partial^2 u}{\partial x \partial y}$

(2) State the order of the given ODE. Determine whether the equation is linear or nonlinear.

1. $(1-x)y''-4xy'+5y=\cos x$

2. $x\frac{d^3y}{dx^3}-\left(\frac{dy}{dx}\right)^4+y=0$

3. $t^5y^{(4)}-t^3y''+6y=0$

4. $\frac{d^2u}{dr^2}+\frac{du}{dr}+u=\cos(r+u)$

5. $\frac{d^2y}{dx^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

6. $\frac{d^2R}{dt^2}=-\frac{k}{R^2}$

7. $(\sin\theta)\,y'''-(\cos\theta)\,y'=2$

8. $(y^2-1)dx+x\,dy=0$

9. $u\,dv+(v+uv-ue^u)\,du=0$

➢ System of Ordinary Differential Equations

A **system of ordinary differential equations** is two or more equations involving the derivatives of two or more unknown functions of a single independent variable. For example, if *x* and *y* denote dependent variables and *t* denotes the independent variable, then a system of two first-order differential equations is given by

$$\frac{dx}{dt}=f(t,x,y)$$
$$\frac{dy}{dt}=g(t,x,y)$$

A **solution** of a system is a pair of differentiable functions$x=\emptyset_1(x),\ y=\emptyset_2(x)$, defined on a common interval *I*, that satisfy each equation of the system on this interval.

1.2 Initial value problems (IVP)

On some interval *I* containing x_0 the problem of solving an n[th] order differential equation subject to *n* side conditions specified at x_0

Solve:

$$\frac{d^n y}{dx^n} = f(x, y, y', y'', \dots \dots, y^{(n-1)})$$

Subject to

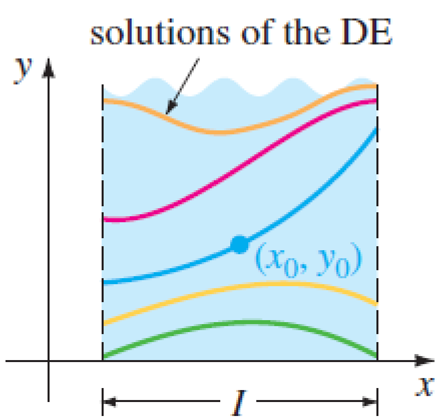
$$y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$$

The value of *y(x)* and its *n-1* derivatives at x_0 are called initial conditions (IC). For example,

(a) Solve: $\frac{dy}{dx} = f(x, y)$

Subject to: $y(x_0) = y_0$

A solution *y*(*x*) of the differential equation $y' = f(x, y)$ on an interval *I* containing x_0 so that its graph passes through the specified point (x_0, y_0). A solution curve is shown in the Figure.

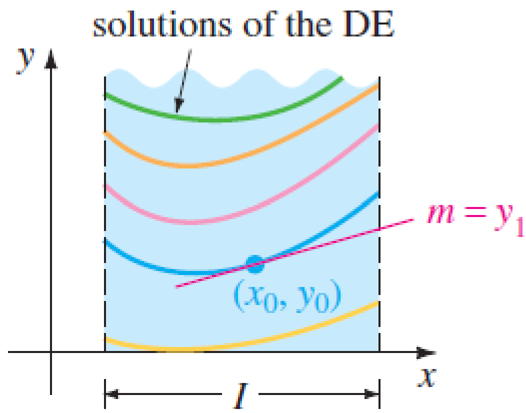


(b) Solve:

$$\frac{d^2 y}{dx^2} = f(x, y, y')$$

Subject to: $y(x_0) = y_0, y'(x_0) = y_1$

To find a solution $y(x)$ of the differential equation $y'' = f(x, y, y')$ on an interval I containing x_0 so that its graph not only passes through (x_0, y_0) but the slope of the curve at this point is the number y_1. A solution curve is shown in the following Figure.



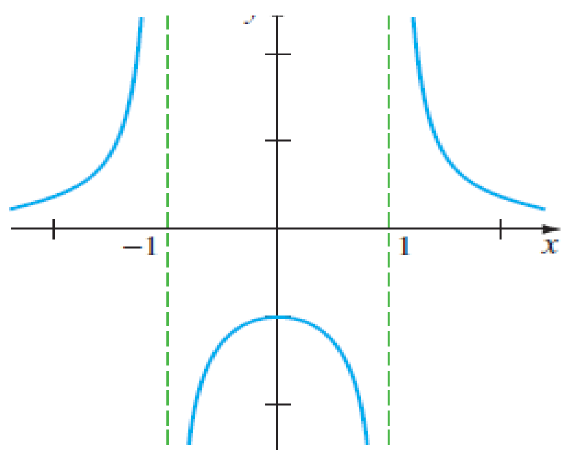
1.2.1 Interval *I* of Definition of a Solution

The one-parameter family of solutions of the first-order differential equation $y' + 2xy^2$ =0 is $y = 1/(x^2 + c)$.

If the initial condition $y(0)$= -1, then substituting into the family of solutions gives $y = 1/(x^2 - 1)$.

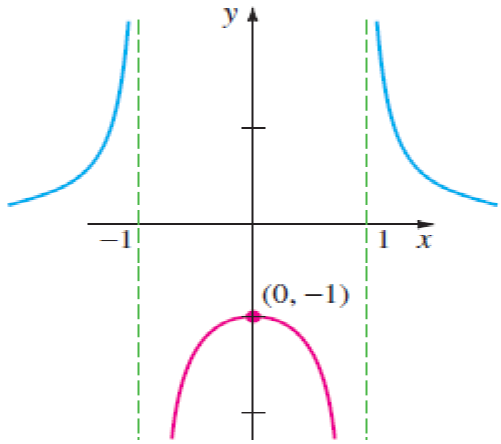
There are three following distinctions:

- Considered as a *function*, the domain of $y = 1/(x^2 - 1)$ is the set of real numbers x for which $y(x)$ is defined; this is the set of all real numbers except x=1 and x=-1.



(a) function defined for all x except $x = \pm 1$

- Considered as a *solution of the differential equation* $y' + 2xy^2 = 0$, the interval *I* of definition of of $y = 1/(x^2 - 1)$ could be taken to be any interval over which *y*(*x*) is defined and differentiable. The largest intervals on which of $y = 1/(x^2 - 1)$ is a solution are $(-\infty,-1)$, $(-1, 1)$, and $(1, \infty)$.



(b) solution defined on interval containing $x = 0$

- Considered as a *solution of the initial-value problem* $y' + 2xy^2 = 0$, $y(0) = -1$, the interval *I* of definition of of $y = 1/(x^2 - 1)$ could be taken to be any interval over which *y*(*x*) is defined, differentiable, *and* contains the initial point $x = 0$; the largest interval for which this is true is $(-1, 1)$.

Examples:

1. Determine values of r, if possible, so that the given differential equation has a solution of the form $y = e^{rx}$, $\quad y'' - 4y = 0$

$$y' = re^{rx}$$
$$y'' = r^2 e^{rx}$$
$$r^2 e^{rx} - 4e^{rx} = 0$$
$$e^{rx}(r^2 - 4) = 0$$
$$e^{rx} \neq 0$$
$$r^2 - 4 = 0$$
$$(r - 2)(r + 2) = 0$$
$$r = \pm 2$$

2. Given $y = C_1 e^{3x} + C_2 e^{-2x}$ is a general solution of the differential equation $y'' - y' - 6y = 0$

Find a solution to the IVP:

$[y'' - y' - 6y = 0,\ y(0) = 2,\ y'(0) = 5]$

$$y(0) = C_1 e^0 + C_2 e^0$$

$$C_1 + C_2 = 2$$

$$y' = 3C_1 e^{3x} - 2C_2 e^{-2x}$$

$$y'(0) = 3C_1 e^0 - 2C_2 e^0$$

$$3C_1 - 2C_2 = 5$$

$$C_1 = 9/5$$

$$C_2 = 1/5$$

$$y = \frac{9}{5} e^{3x} + \frac{1}{5} e^{-2x}$$

Exercises 2:

1. $y = \frac{1}{1 + c_1 e^{-x}}$ is a one parameter family of solutions of the fiest order DE $y' = y - y^2$. Find a solution of the first order IVP consisting of this DE and the given initial condition

(a) *y(0)= -1/3* (b) *y(-1)=2*

2. $x = c_1 \cos t + c_2 \sin t$ is a two parameter family of solutions of the second order DE $x'' + x = 0$. Find a solution of the second order IVP consisting of this DE and the given initial conditions

(a) $x(0) = -1 , x'(0) = 8$

(b) $x(\pi/2) = 0 , x'(\pi/2) = 1$

3. Verify that y=tan (x+c) is a one parameter family of solutions of the DE $y' = 1 + y^2$

1.3 Differential Equations as Mathematical Models

Construction of a mathematical model of a system starts with

- Identification of the variables that are responsible for changing the system (**Level of resolution)** of the model.
- Make a set of reasonable assumptions about the system.

Since the assumptions made about a system frequently involve *a rate of change* of one or more of the variables, the mathematical model may be a differential equation or a system of differential equations.

1.3.1 Conservation equations

The basis for virtually all theoretical process models is the general conservation principle. It is written as an equation form in the following manner of rates (per unit time)

[rate of Accumulation of S] = [Rate of Input of S] – [Rate of Output of S] + [Rate of Production of S]

where, *S* is a conserved quantity within the boundaries of a system.

The basic model structure of a chemical process can be developed making balance in terms of three fundamental quantities:

- Mass

The mass balance equation may be developed with respect to the total mass or to the mass of individual components in a mixture

- Energy

In chemical process modelling, the energy balance equations are also sometimes named as heat balance equations.

- Momentum

The momentum balance equation is a generalization of Newton's law of motion which states that force is the product of mass and acceleration.

1.3.2 Applications

- **Growth and decay**

Problems describing growth are characterized by a positive value of k whereas problems involving decay yield a negative k value. Thus, k is either a growth constant ($k > 0$) or a decay constant ($k < 0$).

- **Population Dynamics**

To model growth of small populations over short intervals of time (for example, bacteria growing in a petri dish).

$$\frac{dP}{dt} \propto P(t) \;\; or \;\; \frac{dP}{dt} = kP(t)$$

where k is a constant of proportionality ($k > 0$)

- **Radioactive Decay**

For example, over time the highly radioactive radium, Ra-226, transmutes into the radioactive gas radon, Rn-222. To model the phenomenon of **radioactive decay,** it is assumed that the rate $\frac{dA}{dt}$ at which the nuclei of a substance decay is proportional to the amount (more precisely, the number of nuclei) $A(t)$ of the substance remaining at time t:

$$\frac{dA}{dt} \propto A(t) \;\; or \;\; \frac{dA}{dt} = kA(t)$$

where k is a constant of proportionality ($k < 0$)

- **Newton's Law of Cooling/Warming**

The rate at which the temperature of a body changes is proportional to the difference between the temperature of the body and the temperature of the surrounding medium. If $T(t)$ represents the temperature of a body at time t, T_m the temperature of the surrounding medium, and $\frac{dT}{dt}$ the rate at which the temperature of the body changes, then Newton's law of cooling/warming translates into the mathematical statement

$$\frac{dT}{dt} \propto T - T_m \;\; or \;\; \frac{dT}{dt} = k(T - T_m)$$

where k is a constant of proportionality. In either case, cooling or warming, if T_m is a constant, it stands to reason that $k < 0$

➢ Chemical Reactions

In chemistry:

➢ First-Order Chemical Reaction

If the molecules of substance A decompose into smaller molecules, the rate at which this decomposition takes place is proportional to the amount of the first substance that has not undergone conversion.

If X(t) is the amount of substance A remaining at any time, then $\frac{dX}{dt} = kX$, where k is a negative constant since X is decreasing.

An example of a first-order chemical reaction is the conversion of t-butyl chloride, $(CH_3)_3CCl$, into t-butyl alcohol, $(CH_3)_3COH$:

$$(CH_3)_3CCl + NaOH \rightarrow (CH_3)_3COH + NaCl$$

Only the concentration of t-butyl chloride controls the rate of reaction.

➢ Second-Order Chemical Reaction

The reaction one molecule of sodium hydroxide, NaOH, is consumed for every molecule of methyl chloride, CH_3Cl, thus forming one molecule of methyl alcohol, CH_3OH, and one molecule of sodium chloride, NaCl.

$$CH_3Cl + NaOH \rightarrow CH_3OH + NaCl$$

In this case the rate at which the reaction proceeds is proportional to the product of the remaining concentrations of CH_3Cl and NaOH.

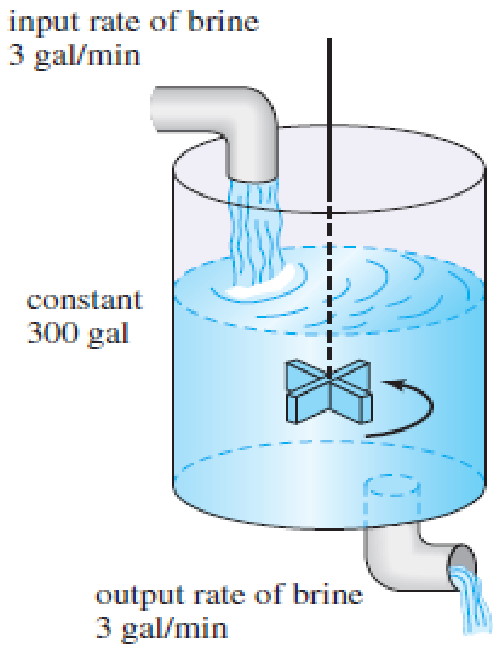
Let us suppose one molecule of a substance A combines with one molecule of a substance B to form one molecule of a substance C. If X denotes the amount of chemical C formed at time t and if $\propto$ and β are, in turn, the amounts of the two chemicals A and B at $t = 0$ (the initial amounts), then the instantaneous amounts of A and B not converted to chemical C are $\propto - X$ and $\beta - X$, respectively. Hence the rate of formation of C is given by

$$\frac{dX}{dt} = k(\propto - X)(\beta - X)$$

- **Mixtures**

The mixing of two salt solutions of differing concentrations gives rise to a first-order differential equation for the amount of salt contained in the mixture.

Let us suppose that a large mixing tank initially holds 300 gallons of brine. Another brine solution is pumped into the large tank at a rate of 3 gallons per minute; the concentration of the salt in this inflow is 2 pounds per gallon. When the solution in the tank is well stirred, it is pumped out at the same rate as the entering solution.



If $A(t)$ denotes the amount of salt (measured in pounds) in the tank at time t, then the rate at which $A(t)$ changes is a net rate:

$$\frac{dA}{dt} = (input\ rate\ of\ salt) - (output\ rate\ of\ salt) = R_{in} - R_{out}$$

The input rate R_{in} at which salt enters the tank is the product of the inflow concentration of salt and the inflow rate of fluid. R_{in} is measured in pounds per minute:

$$R_{in} = \overbrace{(2\ lb/gal)}^{concentration\ of\ salt\ in\ inflow} \cdot \overbrace{(3\ gal/min)}^{input\ of\ brine} = \overbrace{(6\ lb/min)}^{input\ rate\ of\ salt}$$

Since the solution is being pumped out of the tank at the same rate that it is pumped in, the number of gallons of brine in the tank at time t is a constant 300 gallons. Hence the concentration of the salt in the tank as well as in the outflow is

$c(t) = \frac{A(t)}{300}\ lb/gal$, so the output rate R_{out} of salt is

$$R_{out} = \overbrace{\left(\frac{A(t)}{300}\, lb/gal\right)}^{concentration\ of\ salt\ in\ outflow} \cdot \overbrace{(3\ gal/min)}^{outrate\ of\ brine} = \overbrace{\left(\frac{A(t)}{100}\ lb/min\right)}^{input\ rate\ of\ salt}$$

The net rate becomes

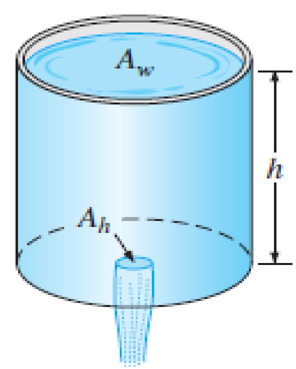
$$\frac{dA}{dt} = 6 - \frac{A}{100} \quad or \quad \frac{dA}{dt} + \frac{A}{100} = 6$$

➢ Draining a Tank

In hydrodynamics, **Torricelli's law** states that the speed *v* of efflux of water though a sharp-edged hole at the bottom of a tank filled to a depth *h* is the same as the speed that a body (in this case a drop of water) would acquire in falling freely from a height *h*—that is, $v = \sqrt{2gh}$, where g is the acceleration due to gravity. This last expression comes from equating the kinetic energy $\frac{1}{2}mv^2$ with the potential energy *mgh* and solving for *v*.

Suppose a tank filled with water is allowed to drain through a hole under the influence of gravity. To find the depth *h* of water remaining in the tank at time *t*. If the area of the hole is A_h (in ft^2) and the speed of the water leaving the tank is $v = \sqrt{2gh}$ (in ft/s), then the volume of water leaving the tank per second is $A_h\sqrt{2gh}$ (in ft3/s). Thus if *V*(*t*) denotes the volume of water in the tank at time *t*, then

$$\frac{dV}{dt} = -A_h\sqrt{2gh}$$



where the minus sign indicates that *V* is decreasing.

Note here that we are ignoring the possibility of friction at the hole that might cause a reduction of the rate of flow there.

Now if the tank is such that the volume of water in it at time t can be written $V(t) = A_w h$, where A_w (in ft^2) is the *constant* area of the upper surface of the water, then

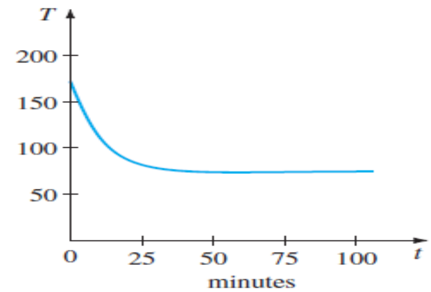
$$\frac{dV}{dt} = A_w \frac{dh}{dt}$$

Substituting in $\frac{dV}{dt} = -A_h\sqrt{2gh}$ gives us the desired differential equation for the height of the water at time t

$$\frac{dh}{dt} = -\frac{A_h}{A_w}\sqrt{2gh}$$

Exercises:

1. A cup of coffee cools according to Newton's law of cooling. Use data from the graph of the temperature $T(t)$ in the following Figure to estimate the constants T_m, T_0, and k in a model of the form of a first-order initial-value problem: $\frac{dT}{dt} = k(T - T_m), T(0) = T_0$



2. Suppose that a large mixing tank initially holds 300 gallons of water in which 50 pounds of salt have been dissolved. Pure water is pumped into the tank at a rate of 3 gal/min, and when the solution is well stirred, it is then pumped out at the same rate. Determine a differential equation for the amount of salt $A(t)$ in the tank at time $t > 0$. What is $A(0)$?

3. Suppose water is leaking from a tank through a circular hole of area A_h at its bottom. When water leaks through a hole, friction and contraction of the stream near the hole reduce the volume of water leaving the tank per second to $cA_h\sqrt{2gh}$ where c $(0 < c < 1)$ is an empirical constant.

Determine a differential equation for the height h of water at time t for the cubical tank shown in the following Figure. The radius of the hole is 2 in., and $g = 32$ ft/s^2.