Chapter 3
Modeling with First-Order Differential Equations

3.1 Linear Models

Example (1) Bacterial Growth

Aculture initially has P_0 number of bacteria. At t =1 h the number of bacteria is measured to be $\frac{3}{2}P_0$, determine the time necessary for the number of bacteria to triple.

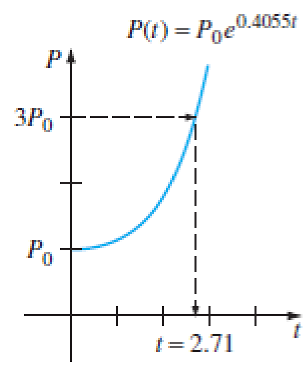
$$\frac{dP}{dt} = kP \quad at\ t_o = 0\ and\ P(0) = P_o\,, \qquad at\ t = 1\ and\ P(1) = \frac{3}{2}P_o$$

$$P(t) = ce^{kt}$$

$$at\ t = 0\ \ P(0) = P_o = ce^o \Longrightarrow c = P_o$$

$$at\ t = 1\ \ P(1) = \frac{3}{2}P_o = P_o e^k \Longrightarrow k = 0.4055$$

$$P(t) = 3P_o = P_o e^{0.4055t} \Longrightarrow t = \frac{\ln 3}{0.4055} = 2.71\ h$$



Example 2 Half-Life of Plutonium

A breeder reactor converts relatively stable uranium-238 into the isotope plutonium- 239. After 15 years it is determined that 0.043% of the initial amount A_0 of plutonium has disintegrated. Find the half-life of this isotope.

$$\frac{dA}{dt} = kA \quad A(0) = A_0$$

$$A(t) = A_0 e^{kt}$$

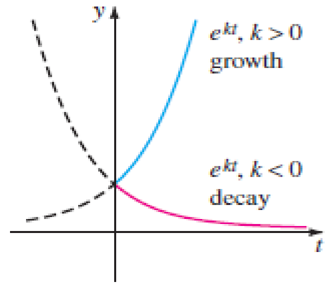
If 0.043% of the atoms of A_0 have disintrgrated, then 99.957% of the substance remains.

$$0.99957A_0 = A_0 e^{15k} \Longrightarrow k = -0.00002867$$

$$A(t) = A_0 e^{-0.00002867t}$$

$$0.5\ A_0 = A_0 e^{-0.00002867t} \Longrightarrow t = 24{,}180\ yr$$

Note that



Growth (k > 0) and decay (k < 0)

Example 3: Age of a Fossil

Afossilized bone is found to contain 0.1% of its original amount of C-14. Determine the age of the fossil. The half-life of C-14 was approximately 5730 years

$$\frac{dA}{dt} = kA$$

$$A(t) = A_0 e^{kt}$$

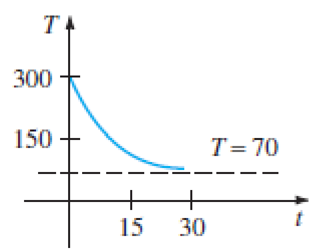
$$\frac{1}{2}A_0 = A_0 e^{5730k} \Longrightarrow k = -0.00012097$$

$$0.001A_0 = A_0 e^{-0.00012097t} \Longrightarrow t = 57{,}100\ yr$$

Example 4: Cooling of a Cake

When a cake is removed from an oven, its temperature is measured at 300° F. Three minutes later its temperature is 200° F. How long will it take for the cake to cool off to a room temperature of 70° F?

$$\frac{dT}{dt} = k(T - 70),\ \ T(0) = 300$$



$$\frac{dT}{T - 70} = kdt$$

$$T = 70 + c_2 e^{kt}$$

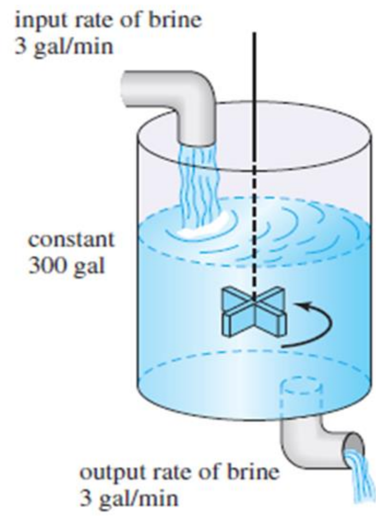
$$T(0) = 300 = 70 + c_2 \Longrightarrow c_2 = 230$$

$$T(3) = 200 = 70 + 230e^{3k} \Longrightarrow k = -0.19018$$

$$T = 70 + 230e^{-0.19018t}$$

Example 5: Mixture of Two Salt Solutions

The large tank holds 300 gallons of a brine solution. Salt was entering and leaving the tank; a brine solution was being pumped into the tank at the rate of 3 gal/min; it mixed with the solution there, and then the mixture was pumped out at the rate of 3 gal/min. The concentration of the salt in the inflow, or solution entering, was 2 lb/gal, If 50 pounds of salt were dissolved initially in the 300 gallons, how much salt is in the tank after a long time?



$$\frac{dA}{dt} = R_{in} - R_{out}$$

$$\frac{dA}{dt} = \left(2\ \frac{lb}{gal}\right)\left(3\ \frac{gal}{\text{min}}\right) - \left(\frac{A(t)}{300}\ \frac{lb}{gal}\right)\left(3\ \frac{gal}{\text{min}}\right)$$

$$\frac{dA}{dt} + \frac{A}{100} = 6, \qquad A(0) = 50lb$$

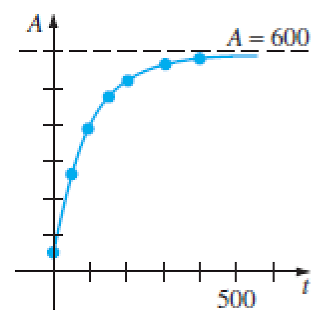
By using integrating factor of the linear differential equation is $e^{t/100}$

$$\frac{d}{dt}\left[e^{t/100}\, A\right] = 6e^{t/100}$$

$$A(t) = 600 + ce^{-t/100}$$

$$A(0) = 50 = 600 + c \Longrightarrow c = -550$$

$$A(t) = 600 - 550\, e^{-t/100}$$



Example 5 Revisited

If the well-stirred solution in Example 5 is pumped out at a slower rate of 2 gal/min. then liquid will accumulate in the tank at the rate of $r_{in} - r_{out} = (3-2)\frac{gal}{min} = 1\ gal/min$. After t minutes,

$$(1\ gal/min)(t\ \text{min}) = t\ gal$$

The tank will contain 300 +t gallons of brine. The concentration of salt

$$\frac{dA}{dt} = R_{in} - R_{out}$$

$$\frac{dA}{dt} = \left(2\ \frac{lb}{gal}\right)\left(3\ \frac{gal}{\min}\right) - \left(\frac{A(t)}{300+t}\ \frac{lb}{gal}\right)\left(2\ \frac{gal}{\min}\right)$$

$$\frac{dA}{dt} - \frac{2A}{300+t} = 6, \qquad A(0) = 50lb$$

The integrating factor is

$$e^{\int 2dt/(300+t)} = e^{2\ln(300+t)} = (300+t)^2$$

and so after multiplying by the factor the equation is cast into the form

$$\frac{d}{dt}[(300+t)^2 A] = 6(300+t)^2$$

$$(300+t)^2 A = 2(300+t)^3 + c\,, A(0) = 50$$

Solving for A

$$A(t) = 600 + 2t - (4.95x10^7)(300+t)^{-2}$$

3.2 Nonlinear Models

Chemical Reactions

Suppose that a grams of chemical A are combined with b grams of chemical B. If there are M parts of A and N parts of B formed in the compound and X(t) is the number of grams of chemical C formed, then the number of grams of chemical A and the number of grams of chemical B remaining at time t are, respectively,

$$a - \frac{M}{M+N}X \quad and \quad b - \frac{N}{M+N}X$$

The law of mass action states that when no temperature change is involved, the rate at which the two substances react is proportional to the product of the amounts of A and B that are untransformed (remaining) at time t:

$$\frac{dX}{dt} \propto \left(a - \frac{M}{M+N}X\right)\left(b - \frac{N}{M+N}X\right)$$

It factor out M/(M+N) from the first factor and N/(M+N) from the second factor

$$\frac{dX}{dt} = k\,(\alpha - X)(\beta - X)$$

$$\alpha = \frac{a(M+N)}{M} \quad and\ \beta = \frac{b(M+N)}{N}$$

Example 2: Second-Order Chemical Reaction

A compound C is formed when two chemicals A and B are combined. The resulting reaction between the two chemicals is such that for each gram of A, 4 grams of B is used. It is observed that 30 grams of the compound C is formed in 10 minutes.Determine the amount of C at time t if the rate of the reaction is proportional to the amounts of A and B remaining and if initially there are 50 grams of A and 32 grams of B. How much of the compound C is present at 15 minutes? Interpret the solution **as $t \to \infty$.**

$$\frac{dX}{dt} = k\,(a - X)(\beta - X)$$

$$\alpha = (50)\frac{(1+4)}{1} = 250\,, \beta = (32)\frac{(1+4)}{4} = 40$$

$$\frac{dX}{dt} = k\,(250 - X)(40 - X)$$

By separation of variables and partial fractions

$$\frac{-\frac{1}{210}}{(250-X)}dX + \frac{\frac{1}{210}}{(40-X)}dX = k\,dt$$

$$\ln\frac{(250-X)}{(40-X)} = 210kt + c_1$$

$$\frac{(250-X)}{(40-X)} = c_2\, e^{210kt}$$

At $t = 0, X = 0 \;\to c_2 \;= \frac{25}{4}$

At $t = 10, X = 30 \;\to k = 0.1258$

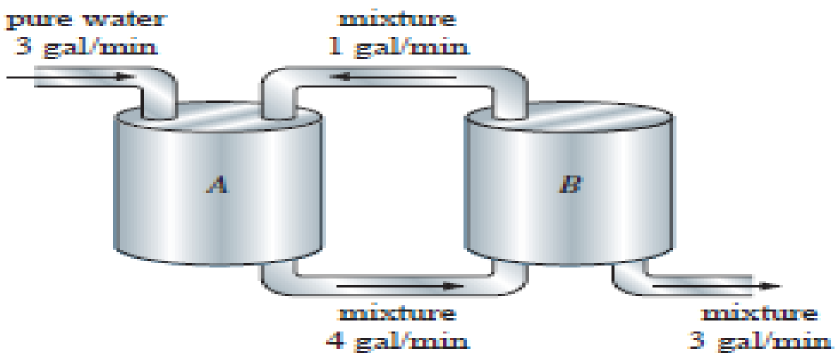
$$X(t) = 1000\,\frac{1 - e^{-0.1258t}}{25 - 4e^{-0.1258t}}$$

$$X(15) = 34.78\ grams$$

$$\lim_{t\to\infty} X(t) = 40\ grams$$

3.3 Modeling With Systems Of First-Order Des

Mixtures Consider the two tanks: tank *A* contains 50 gallons of water in which 25 pounds of salt is dissolved. Suppose tank *B* contains 50 gallons of pure water. Liquid is pumped into and out of the tanks as indicated in the following figure; the mixture exchanged between the two tanks and the liquid pumped out of tank *B* are assumed to be well stirred. To construct a mathematical model that describes the number of pounds $x_1(t)$ and $x_2(t)$ of salt in tanks *A* and *B*, respectively, at time *t*.



$$\frac{dx_1}{dt} = \overbrace{\left(\frac{3gal}{min}\right)\left(0\frac{lb}{gal}\right) + \left(\frac{1gal}{min}\right)\left(\frac{x_2}{50}\frac{lb}{gal}\right)}^{\text{Input rate of salt}} - \overbrace{\left(\frac{4gal}{min}\right)\left(\frac{x_1}{50}\frac{lb}{gal}\right)}^{\text{Output rate of salt}}$$

$$\frac{dx_1}{dt} = -\frac{2}{25}x_1 + \frac{1}{50}x_2$$

$$\frac{dx_1}{dt} = \overbrace{\left(\frac{4gal}{min}\right)\left(\frac{x_1}{50}\frac{lb}{gal}\right)}^{\text{Input rate of salt}} - \overbrace{\left(\frac{3gal}{min}\right)\left(\frac{x_2}{50}\frac{lb}{gal}\right) + \left(\frac{1gal}{min}\right)\left(\frac{x_2}{50}\frac{lb}{gal}\right)}^{\text{Output rate of salt}}$$

$$\frac{dx_2}{dt} = \frac{2}{25}x_1 - \frac{2}{25}x_2$$

The linear system

$$\frac{dx_1}{dt} = -\frac{2}{25}x_1 + \frac{1}{50}x_2$$

$$\frac{dx_2}{dt} = \frac{2}{25}x_1 - \frac{2}{25}x_2$$