Chapter -4-

Higher-Order Differential Equations

5.1 Initial-Value and Boundary-Value Problems

- **Initial-Value Problem (IVP)**

For a linear differential equation an ***n*th-order initial-value problem** is

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$
$$y(x_0)=y_0, y'(x_0)=y_1,\cdots,y^{(n-1)}(x_0)=y_{n-1}$$

The solution is a function defined on some interval I, containing x_0 that satisfies the differential equation and the n initial conditions specified at x_0

$$y(x_0)=y_0, y'(x_0)=y_1,\cdots,y^{(n-1)}(x_0)=y_{n-1}$$

In the case of second order initial value problem a solution curve must pass through the point (x_0, y_0) and have slope y_1 at this point

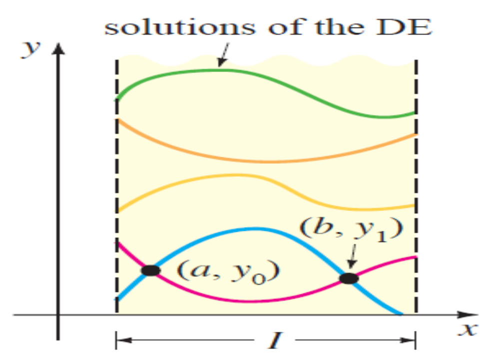
- **Boundary-Value Problem (BVP)**

Boundary – value problem consists of solving a linear differential equation of order two or greater in which the dependent variable y or its derivatives are specified at different points. A problem such as

$$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$
$$y(a)=y_0, y(b)=y_1$$

$y(a)=y_0, y(b)=y_1$ are called boundary conditions. A solution of the foregoing problem is a function satisfying the differential equation on some interval I, containing a and b, whose graph passes through the two points (a, y_0) and (b, y_1). For a second-order differential equation other pairs of boundary conditions could be

$$y(a)=y_0, y'(b)=y_1$$
$$y'(a)=y_0, y(b)=y_1$$
$$y'(a)=y_0, y'(b)=y_1$$



Example 1: A BVP Can Have Many, One, or No Solutions

The two-parameter family of solutions of the differential equation

$$x'' + 16x = 0, x = c_1 cos4t + c_2 sin4t$$

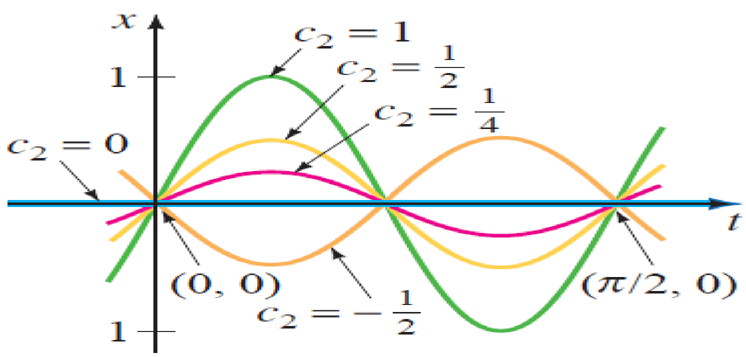
(a) To determine the solution of the equation that further satisfies the boundary conditions $x(0) = 0\,, x\left(\frac{\pi}{2}\right) = 0$.

Observe that the first condition implies that $c_1 = 0$, so $x = c_2 \sin 4t$. But when $t = \frac{\pi}{2}$ is satisfied for any choice of c_2, since $\sin 2\pi = 0$ Hence the boundary-value problem

$$x'' + 16x = 0, \qquad x(0) = 0\,, x\left(\frac{\pi}{2}\right) = 0$$

has infinitely many solutions.

The following Figure shows the graphs of some of the members of the one-parameter family $x = c_2 \sin 4t$ that pass through the two points (0, 0) and $(\frac{\pi}{2}, 0)$.



(b) If the boundary-value problem is changed to

$$x'' + 16x = 0, \qquad x(0) = 0\,, x\left(\frac{\pi}{8}\right) = 0$$

then $x(0) = 0$ still requires $c_1 = 0$, by applying $x(\frac{\pi}{8}) = 0$ demands that $0 = c_2 \sin \frac{\pi}{2} = c_2$. 1so $c_2 = 0$. Hence $x = 0$ is the only solution of this new boundary-value problem.

(c) If the boundary-value problem is changed to

$$x'' + 16x = 0, \qquad x(0) = 0\,, x\left(\frac{\pi}{2}\right) = 1$$

Again from $x(0) = 0$ that $c_1 = 0$, but applying $x(\frac{\pi}{2}) = 1$ to $x = c_2 \sin 4t$ leads to the contradiction $1 = c_2 \sin 2\pi$, $1 = c_2\,.\,0$ Hence the boundary-value problem has no solution.

5.2 Homogeneous Equations

A linear *n*th-order differential equation of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

is said to be **homogeneous,** whereas an equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

with *g*(*x*) not identically zero, is said to be **nonhomogeneous.**

For example,

$2y'' + 3y' - 5y = 0$ is a homogeneous linear second-order differential equation

$x^3y''' + 6y' + 10y = e^x$ is a nonhomogeneous linear third-order differential equation.

5.3 Differential Operators

In calculus differentiation is often denoted by the capital letter *D* that is, *dy*/*dx* =*Dy*. For example,

$D(\cos 4x) = -4\sin 4x$ and $D(5x^3 - 6x^2) = 15x^2 - 12x$.

Higher-order derivatives can be expressed in terms of *D* in a natural manner:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D(Dy) = D^2y \qquad and\ \frac{d^ny}{dx^n} = D^ny$$

where *y* represents a sufficiently differentiable function.

For example,

$$y'' + 5y' + 6y = 5x - 3, D^2y + 5Dy + 6y = 5x - 3\ or\ (D^2 + 5D + 6)y = 5x - 3$$

5.4 Superposition Principle

The sum, or superposition, of two or more solutions of a homogeneous linear differential equation is also a solution. Let $y_1, y_2, \ldots, y_k$ be solutions of the homogeneous nth-order differential equation on an interval I. Then the linear combination

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x)$$

where the c_i, i= 1, 2, . . . , k are arbitrary constants, is also a solution on the interval.

Notice

A. A constant multiple $y = c_1 y_1(x)$ of a solution $y_1(x)$ of a homogeneous linear differential equation is also a solution.
B. A homogeneous linear differential equation always possesses the trivial solution y = 0.
C. If a set of two functions is linearly dependent, then one function is simply a constant multiple of the other. For example, the set functions $f_1(x) = \sin 2x$, $f_2(x) = \sin x \ \cos x$
D. A set of two functions is linearly independent when neither function is a constant multiple of the other.

5.5 Linear Dependence and Linear Independence

- **Linear Dependence**

A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is said to be **linearly dependent** on an interval *I* if there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for every *x* in the interval.

If a set of two functions is linearly dependent then one function is a constant multiple of the other. For example, $f_1(x) = \sin 2x$, $f_2(x) = \sin x \cos x$, $\sin 2x = \mathbf{2} \sin x \cos x$ on interval (-∞, ∞)

- **Linear Independence**

If the set of functions is not linearly dependent on the interval, it is said to be **linearly independent** for every *x* in the interval when $c_1 = c_2 = \ldots = c_n = 0$

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

For example, $f_1(x) = x$, $f_2(x) = |x|$ on interval (-∞, ∞)

Example 2: Determine whether the given set of functions is linearly independent or linearly dependent

1. The set functions $f_1(x) = cos^2x$, $f_2(x) = sin^2x, f_3(x) = sec^2x, f_4(x) = tan^2x$

$$c_1 cos^2x + c_2 sin^2x + c_3 sec^2x + c_4 tan^2x = 0$$

Since

$cos^2x + sin^2x = 1 \ and \ \ 1 + tan^2x = sec^2x \rightarrow -sec^2x + tan^2x = -1$

Then

$$cos^2x + sin^2x - sec^2x + tan^2x = 0$$

Compare

$$c_1 cos^2 x + c_2 sin^2 x + c_3 sec^2 x + c_4 tan^2 x = 0$$

$$c_1 = c_2 = 1, c_3 = -1 \; and \; c_4 = 1$$

The set is linearly depentent on the interval (-π/2 ,π/2)

2. The set functions $f_1(x) = \sqrt{x} + 5\,, f_2(x) = \sqrt{x} + 5x, f_3(x) = x - 1, f_4(x) = x^2$

$$c_1(\sqrt{x} + 5) + c_2(\sqrt{x} + 5x) + c_3(x - 1) + c_4\, x^2 = 0$$

$$(c_1 + c_2)\sqrt{x} + (5c_1 - c_3) + (5c_2 + c_3)x + c_4\, x^2 = 0$$

$$c_1 + c_2 = 0 \rightarrow c_1 = -c_2$$

$$5c_1 - c_3 = 0 \rightarrow c_3 = 5c_1$$

$$c_4 = 0$$

Let

$$c_1 = 1, c_2 = -1, c_3 = 5 \; and \; c_4 = 0$$

$$1.(\sqrt{x} + 5) + (-1).(\sqrt{x} + 5x) + 5\,.(x - 1) + 0.\; x^2 = 0$$

The set is linearly depentent on the interval (0 , ∞)

3. The set functions $f_1(x) = 0\,, f_2(x) = x, f_3(x) = e^x$ since $f_1(x) = 0$ the set is linear dependent

$$c. f_1(x) + 0. f_2(x) + 0. f_3(x) = 0$$

Examples:

1. The given family of functions is the general solution of the differential equation on the indicated interval. Find a member of the family that is a solution of the initial value problem.

(a) $y = c_1 + c_2 \cos x + c_3 \sin x\,, (-\infty, \infty)$

$$y'' + y' = 0, y(\pi) = 0, y'(\pi) = 2, y''(\pi) = -1$$

$$y(\pi) = 0 = c_1 + c_2 \cos \pi + c_3 \sin \pi \rightarrow c_1 - c_2 = 0$$

$$y'(\pi) = 2 = -c_2 \sin \pi + c_3 \cos \pi \rightarrow c_3 = -2$$

$$y''(\pi) = -1 = -c_2 \cos \pi + 2 \sin \pi \rightarrow c_2 = -1 \rightarrow c_1 = -1$$

$$y = -1 - \cos x - 2 \sin x$$

(b) $y = c_1 + c_2 x^2$, (-∞, ∞)

$$xy'' - y' = 0, y(0) = 0, y'(0) = 1$$

$$y(0) = 0 = c_1 + c_2.0 \rightarrow c_1 = 0$$

$$y'(0) = 1 = c_2.0 \text{ contradiction}$$

No solution

(c) $y = c_1 + c_2x^2$, (-∞, ∞)

$$xy'' - y' = 0, y(0) = 0, y'(0) = 0$$

$$y(0) = 0 = c_1 + c_2.0 \rightarrow c_1 = 0$$

$$y'(0) = 0 = c_2.0 \rightarrow c_2 \; is \; arbitrary$$

Many solutions such as $y = x^2, y = 2x^2$

Exercises

1. Given that $x(t) = c_1 \cos wt + c_2 \sin wt$ is the general solution of $x'' + w^2x = 0$ on the interval (-∞, ∞), show that

(a) a solution satisfying the initial conditions $x(0) = x_0$, $x'(0) = x_1$ is given by

$$x(t) = x_0 \cos wt + \frac{x_1}{w} \sin wt$$

(b) a solution satisfying the initial conditions $x(t_0) = x_0$, $x'(t_0) = x_1$ is given by

$$x(t) = x_0 \cos w(t - t_0) + \frac{x_1}{w} \sin w(t - t_0)$$

2. Find the member of a family solution $y = c_1e^x + c_2e^{-x}$ of $y'' - y = 0$ that satisfies the boundary conditions $y(0) = 0$, $y(1) = 1$.

3. Determine whether the given set of functions is linearly independent or linearly dependent on the interval (-∞, ∞) and identify the constants

1	$f_1(x) = x , f_2(x) = x^2, f_3(x) = 4x - 3x^2$
2	$f_1(x) = 5 , f_2(x) = cos^2x , f_3(x) = sin^2x$
3	$f_1(x) = \cos 2x \; , f_2(x) = 1, f_3(x) = cos^2x$
4	$f_1(x) = x , f_2(x) = x - 1, f_3(x) = x + 3$
5	$f_1(x) = x , f_2(x) = x^2, f_3(x) = 4x - 3x^2$
6	$f_1(x) = 2 + x , f_2(x) = 2 + \lvert x \rvert$

7	$f_1(x) = 2 + x\ , f_2(x) = 2 + \|x\|, f_3(x) = x$
8	$f_1(x) = 1 + x\ , f_2(x) = x, f_3(x) = x^2$
9	$f_1(x) = e^x\ , f_2(x) = e^{-x}, f_3(x) = \sinh x$
10	$f_1(x) = e^{x+2}\ , f_2(x) = e^{x-3}$

5.6 Reduction of Order

Suppose that y_1 denotes a nontrivial solution of $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ and that y_1 is defined on an interval *I*. A second solution y_2 so that the set consisting of y_1 and y_2 is linearly independent on *I*.

$$\frac{y_2(x)}{y_1(x)} = u(x) \rightarrow y_2(x) = u(x)y_1(x)$$

The function $u(x)$ can be found by substituting into the given differential equation. This method is called **reduction of order** because we must solve a linear first-order differential equation to find *u*.

1. Divide $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ by$a_2(x)$ to put in the standard form

$y'' + P(x)y' + Q(x)y = 0\ \ P(x) and\ Q(x)$ are continupous in some interval I.

2. $y_1(x)$is a known solution of DE on I and $y_1(x) \neq 0\ for\ every\ x$ in the interval
3. Define $y = u(x)y_1(x)$
4. A second solution is

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$$

Example: The function $y_1 = x^2$ is a solution of $x^2\, y'' - 3xy' + 4y = 0$. Find the general solution of the differential equation on the interval (0,∞)

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$$

$$P(x) = -\frac{3}{x}\ and\ Q(x) = \frac{4}{x^2}$$

$$y_2 = x^2 \int \frac{e^{3\int \frac{dx}{x}}}{x^4} dx \rightarrow y_2 = x^2 \int \frac{x^3}{x^4} dx$$

$$y_2 = x^2 \ln x$$

$$y = c_1 y_1 + c_2 y_2 \rightarrow y = c_1 x^2 + c_2 x^2 \ln x$$

Example: Given that $y_1 = e^x$ is a solution of $y'' - y = 0$ on the interval (-∞,∞), use reduction of order to find a second solution y_2 and the general solution

$$y'' - y = 0 \rightarrow P(x) = 0 \; and \; Q(x) = -1$$

$$y_2 = e^x \int \frac{e^{\int 0\, dx}}{e^{2x}} dx \rightarrow y_2 = -\frac{1}{2} e^x . e^{-2x}$$

$$y_2 = -\frac{1}{2} e^{-x}$$

$$y = c_1 e^x - \frac{c_2}{2} e^{-x}$$

5.7 Homogeneous Linear Equations with Constant Coefficients

For solving a homogeneous linear *n*th-order differential equation with constant coefficients of the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0$$

where the coefficients $a_i, i = 0,1, \cdots, n$ are real constants and $a_n \neq 0$

By considering the special case of the second order equation (Auxiliary Equation)

The auxiliary equation of the second order differential equation

$$a\, y'' + by' + cy = 0 \; a, b \; and \; c \; are \; constants$$

$$a\, m^2 + bm + c = 0$$

Since the two roots are $m_1 = \frac{(-b + \sqrt{b^2 - 4ac})}{2a}$ $\; and \; m_2 = \frac{(-b - \sqrt{b^2 - 4ac})}{2a}$

There will be three forms of the general solution corresponding to the three cases:

- $m_1 \; and \; m_2 \; are \; real \; and \; distinct \; (b^2 - 4ac > 0)$

- m_1 *and* m_2 *are real and real* $(b^2 - 4ac = 0)$
- m_1 *and* m_2 *are conjugate complex numbers* $(b^2 - 4ac < 0)$

Case I: Distinct Real Roots

Under the assumption that the auxiliary equation has two unequal real roots m_1 and m_2, we find two solutions $y_1 = e^{m_1 x}$ *and* $y_2 = e^{m_2 x}$, and We see that these functions are linearly independent on (-∞, ∞) . It follows that the general solution of (2) on this interval is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Case II: Repeated Real Roots

When $m_1 = m_2$, we necessarily obtain only one exponential solution, $y_1 = e^{m_1 x}$ where $m_1 = -\frac{b}{2a}$. The second solution of the equation is $y_2 = xe^{m_1 x}$. The general solution is

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$

Case III: Conjugate Complex Roots

If m_1 and m_2 are complex, then we can write $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$. The general solution is

$$y = c_1 e^{(\alpha + i\beta\ )x} + c_2 e^{(\alpha + i\beta\ )x} \rightarrow y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

- **Two Equations worth knowing**
  - Equation 1

$$y'' + k^2 y = 0 \rightarrow\ m^2 + k^2 = 0 \rightarrow m_{1,2} = \pm ki, \qquad k \text{ is real number}$$

The general solution is

$$y = c_1 \cos kx + c_1 \sin kx$$

  - Equation 2

$$y'' - k^2 y = 0 \rightarrow\ m^2 - k^2 = 0 \rightarrow m_{1,2} = \pm k$$

The general solution is

$$y = c_1 e^{kx} + c_2 e^{-kx}$$

Since $\cosh kx$ *and* $\sinh kx$ are linearly independent on any interval of the x-axis,

$y = \frac{1}{2}(e^{kx} + e^{-kx}) = \cosh kx, y = \frac{1}{2}(e^{kx} - e^{-kx}) = \sinh kx$

An alternative form for the general solution of

$$y = c_1 \sinh kx + c_2 \cosh kx$$

Example: Solve the following differential equations

(a) $2\,y'' - 5y' - 3y = 0$

$$2\,m^2 - 5m - 3 = 0 \rightarrow m_1 = -\frac{1}{2}, \qquad m_2 = 3$$

$$y = c_1 e^{-\frac{1}{2}x} + c_2 e^{3x}$$

(b) $y'' - 10y' + 25y = 0$

$$m^2 - 10m + 25 = 0 \rightarrow (m-5)^2 = 0 \rightarrow \ m_1 = m_2 = 5$$

$$y = c_1 e^{5x} + c_2 x e^{5x}$$

(c) $y'' + 4y' + 7y = 0$

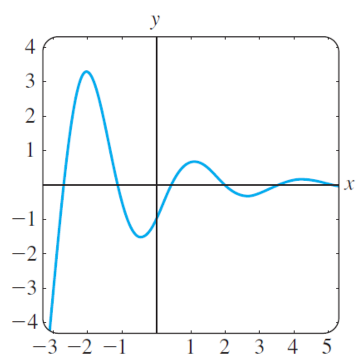
$$m^2 + 4m + 7 = 0 \rightarrow \ m_1 = -2 + \sqrt{3}i\,, m_2 = -2 - \sqrt{3}\,i$$

$$y = e^{-2x}\left(c_1 \cos\sqrt{3}\,x + c_2 \sin\sqrt{3}x\right)$$

Example: Solve IVP

$$4y'' + 4y' + 17y = 0, y(0) = -1, \qquad y'(0) = 2$$

$$4m^2 + 4m + 17 = 0 \rightarrow \ m_1 = -\frac{1}{2} + 2i\,, m_2 = -\frac{1}{2} -$$



$$y = e^{-\frac{x}{2}}(c_1 \cos 2\,x + c_2 \sin 2x)$$

$$y(0) = -1 = 1.(c_1.1 + c_2.0) \rightarrow c_1 = -1$$

$$y' = e^{-\frac{x}{2}}\left((\frac{1}{2} + 2c_2)\cos 2\,x + (2 - \frac{1}{2}c_2)\sin 2x\right)$$

$$y'(0) = 2 \ = \ 1.\left(\left(\frac{1}{2} + 2c_2\right).1 + \left(2 - \frac{1}{2}c_2\right).0\right) \rightarrow c_2 = \frac{3}{4}$$

$$y = e^{-\frac{x}{2}}\left(\frac{3}{4}\sin 2x - \cos 2\,x\right) \quad x \rightarrow \infty\ y \rightarrow 0$$

5.8 Higher-Order Equations

In general, to solve an nth-order differential equation where the a_i, i = 0, 1, . . . , n are real constants, we must solve an nth-degree polynomial equation

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_2 m^2 + a_1 m + a_0 = 0$$

Case I: If all the roots of are real and distinct, then the general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

Case II: When m_1 is a root of multiplicity k of an nth-degree auxiliary equation (k roots are equal to m_1), the general solution must contain the linear combination

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x} + c_3 x^2 e^{m_1 x} + \cdots + c_k x^{k-1} e^{m_1 x}$$

Case III: Complex roots of an auxiliary equation always appear in conjugate pairs. when the auxiliary equation has repeated complex roots the general solution must contain the linear combination

$$y = e^{\alpha x}\Big((c_1 \cos\beta x + c_2 \sin\beta x) + x(c_3 \cos\beta x + c_4 \sin\beta x) + \cdots$$
$$+ x^{k-1}(c_{2k-1} \cos\beta x + c_{2k} \sin\beta x)\Big)$$

Example: Solve

$$y''' + 3y'' - 4y = 0$$

$$m^3 + 3m^2 - 4 = 0$$

From exspection $m_1 = 1$

By division $m^3 + 3m^2 - 4 = (m-1)(m^2 + 4m + 4) = (m-1)(m+2)^2$

$$m_1 = 1, m_{2,3} = -2$$

The general solution

$$y = c_1 e^{x} + c_2 e^{-2x} + c_3 x e^{-2x}$$

Example: Solve

$$y^{(4)} + 2y'' + y = 0$$

$$m^4 + 2m^2 + 1 = (m^2 + 1)^2 = 0$$

$$m_{1,2} = \pm i \, , m_{3,4} = \pm i$$

$$y = (c_1 \cos x + c_2 \sin x) + x(c_3 \cos x + c_4 \sin x)$$

Exercises

1. Each Figure represents the graph of a particular solution of one of the following differential equations:

(a) $y'' - 3y' - 4y = 0$

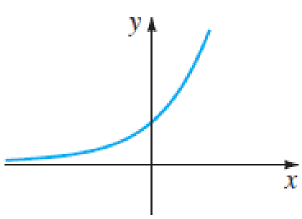
(b) $y'' + 2y' + y = 0$

(c) $y'' + y = 0$

(d) $y'' + 2y' + 2y = 0$

(e) $y'' - 3y' + 2y = 0$

Match a solution curve with one of the differential equations.



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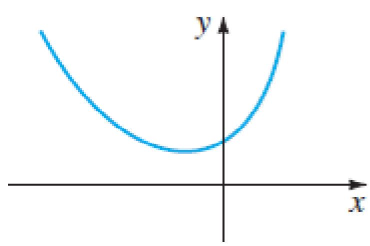

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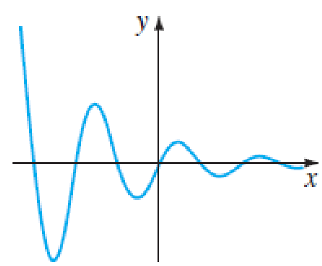

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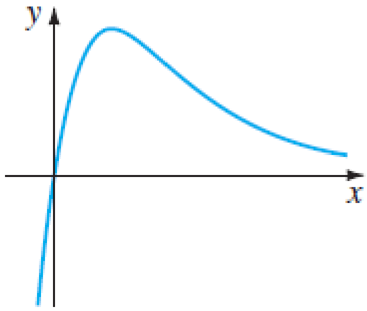

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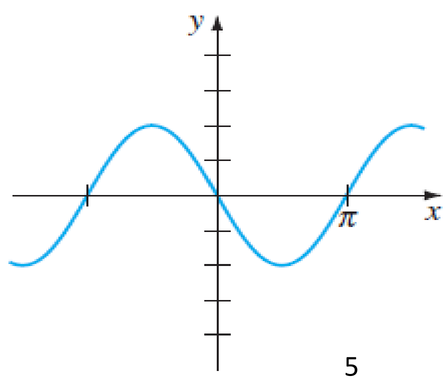

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2. Find a homogeneous linear differential equation with constant coefficients whose general solution is given

(a) $y = c_1 e^{x} + c_2 e^{5x}$

(b) $y = c_1 e^{-4x} + c_2 e^{-3x}$

(c) $y = c_1 + c_2 e^{2x}$

(d) $y = c_1 e^{10x} + c_2 x e^{10x}$

(e) $y = c_1 \cos 3x + c_2 \sin 3x$

(f) $y = c_1 sinh7x + c_2 cosh7x$

(g) $y = c_1 e^{-x} cosx + c_2 e^{-x} sinx$

(h) $y = c_1 + c_2 e^{-2x} cos5x + c_2 e^{-2x} sin5x$

(i) $y = c_1 + c_2 x + c_3 e^{8x}$

(j) $y = c_1 \cos x + c_2 sinx + c_3 cos2x + c_4 sin2x$