Chapter 1 - Lecture 1

Polynomial Interpolation

1.1 Introduction

For the statement

$$y = f(x), x_1 \leq x \leq x_N$$

Every value of x in the range $x_1 \leq x \leq x_N$, there exists one or more values of y. Assuming that *f(x)* is single – valued and continuous and known explicitly, then the values of *f(x)* corresponding to certain given values of *x* can easily be computed and tabulated.

The central problem of numerical analysis is the converse one:

If the values of a function *f (x)* at a set of points $x_1, x_2, ..., x_N$ is known but the analytic expression of the function is unknown that lets us calculate its value at an arbitrary point.

The task now is to estimate *f*(*x*) for arbitrary *x* by drawing a smooth curve through (and perhaps beyond) the x_i.

The procedure of estimating the value of *f (x)* for the desired *x*

- If the value for points between the largest and smallest of the x_i's ($x \in [x_1, x_N]$) is called interpolation
- If the value is for points outside that range ($x \notin [x_1, x_N]$) is called extrapolation.

For example, given the set of tabular values of a census of the population of the Us that is taken every 10 years

Year	Population (in thousands)
1940	132,165
1950	151,326
1960	179,323
1970	203,302
1980	226,542
1990	249,633

To know the population of the US in year 1965 or year 2010, have to fit a function through the given data.

The form of the function ($\phi\ (x)$) that approximates the set of points should be a convenient one and should be applicable to a general class of problems. The most usual class of functions fitted through data are polynomials.

1.2 Polynomial Interpolation

The general formula for an *(n* – 1*)* th-order polynomial can be written as

$$f(x) = a_1 + a_2x + a_3x^2 + \cdots + a_nx^{n-1}$$

For n data points, there is one and only one polynomial of order *(n* − 1*)* that passes through all the points. For example, there is only one straight line (i.e., a first-order polynomial) that connects two points (Fig. 1a). Similarly, only one parabola connects a set of three points (Fig. 1b).

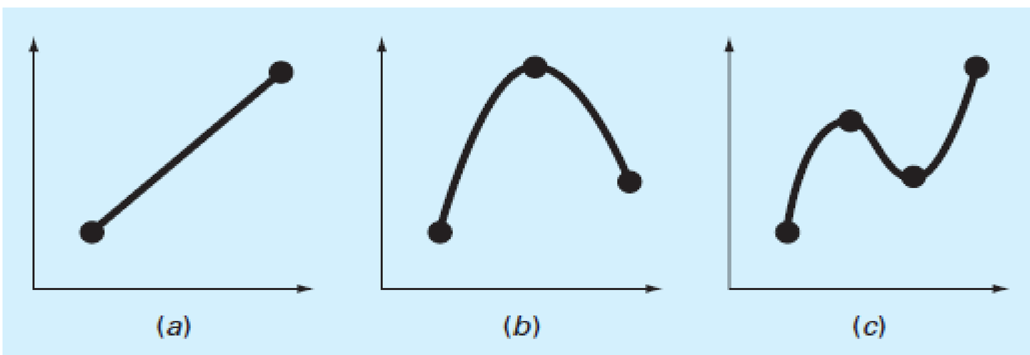


Figure 1: examples of interpolating polynomials: (*a*) first-order (linear) connecting two points, (*b*) second-order (quadratic or parabolic) connecting three points, and (*c*) third-order (cubic) connecting four points.

Polynomial interpolation consists of determining the unique *(n* − 1*)*th order polynomial that fits *n* data points. Then this polynomial provides a formula to compute intermediate values.

1.3 Newton Interpolating Polynomial

Newton's interpolating polynomial is among the most popular and useful forms for expressing an interpolating polynomial. First, introduce the first- and second-order versions because of their simple visual interpretation, before presenting the general equation

1.4.1 Linear Interpolation

The simplest form of interpolation is to connect two data points with a straight line. This technique, called *linear interpolation*, is depicted graphically in Figure 2. The shaded areas indicate the similar triangles used to derive the Newton linear-interpolation formula

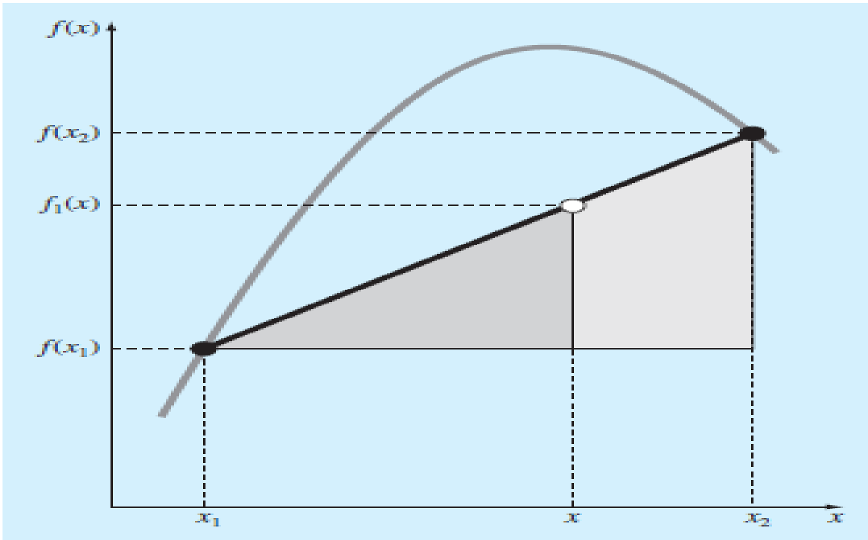


Figure 2: linear-interpolation

$$\frac{f_1(x) - f(x_1)}{x - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Which can be rearranged to

$$f_1(x) = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$$

which is the *Newton linear-interpolation formula*. The notation $f_1(x)$designates that this is a first order interpolating polynomial. Notice that besides representing the slope of the line connecting the points, the term $\frac{f(x_2)-f(x_1)}{x_2-x_1}$ is a finite-difference approximation of the first derivative.

In general, the smaller the interval between the data points, the better the approximation. This is due to the fact that, as the interval decreases, a continuous function will be better approximated by a straight line. This characteristic is demonstrated in the following example.

Example 1: Estimate the natural logarithm of 2 using linear interpolation during interval 1 to 6. Then, repeat the procedure, but use a smaller interval from 1 to 4. Note that the true value of ln 2 is 0.6931472.

For interval [1, 6],

$$f(x_1) = \ln 1 = 0,$$

$$f(x_2) = \ln 6 = 1.791759$$

$$f_1(x) = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$$

$$f_1(2) = \ln 1 + \frac{\ln 6 - \ln 1}{6 - 1}(2 - 1) \rightarrow f_1(2) = 0 + \frac{1.791759 - 0}{5} = 0.3583519$$

which represents an error of $\varepsilon_t = 48.3\%$

For interval [1, 4],

$$f(x_1) = \ln 1 = 0,$$

$$f(x_2) = \ln 4 = 1.386294$$

$$f_1(2) = \ln 1 + \frac{\ln 4 - \ln 1}{4 - 1}(2 - 1) \rightarrow f_1(2) = 0 + \frac{1.386294 - 0}{5} = 0.4620981$$

Thus, using the shorter interval reduces the percent relative error to $\varepsilon_t = 33.3\%$. Both interpolations are shown in Figure 3, along with the true function.

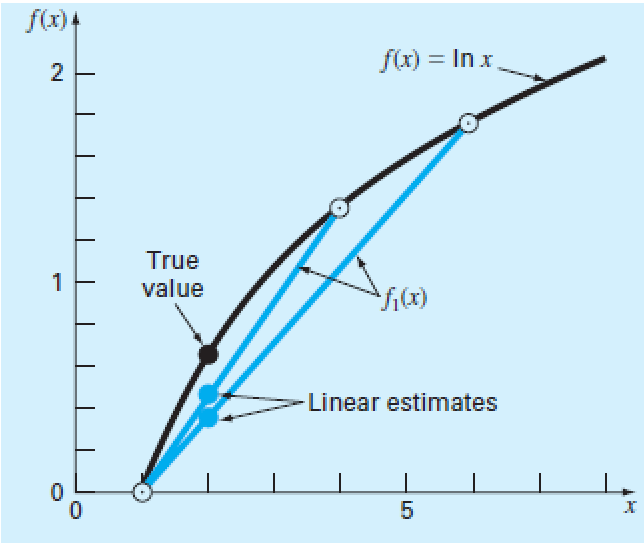


Figure 3: Two linear interpolations to estimate ln 2.

1.4.2 Quadratic Interpolation

If three data points are available, this can be accomplished with a second-order polynomial (also called a quadratic polynomial or a parabola). A particularly convenient form for this purpose is

$$f_2(x) = b_1 + b_2(x - x_1) + b_3(x - x_1)(x - x_2)$$

A simple procedure can be used to determine the values of the coefficients.

x_1	$f(x_1)$		
x_2	$f(x_2)$	$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$	
x_3	$f(x_3)$	$\frac{f(x_3) - f(x_2)}{x_3 - x_2}$	$\frac{\frac{f(x_3) - f(x_2)}{x_3 - x_2} - \frac{f(x_2) - f(x_1)}{x_2 - x_1}}{x_3 - x_1}$

$$b_1 = f(x_1)$$

$$b_2 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$b_3 = \frac{\frac{f(x_3) - f(x_2)}{x_3 - x_2} - \frac{f(x_2) - f(x_1)}{x_2 - x_1}}{x_3 - x_1}$$

Example 2: Employ a second order Newton polynomial to estimated ln 2 with

$x_1 = 1$	$f(x_1) = 0$	0.4620981	−0.051731
$x_2 = 4$	$f(x_2) = 1.386294$	0.2027325	
$x_3 = 6$	$f(x_3) = 1.791759$		

$$b_1 = 0$$

$$b_2 = \frac{1.386294 - 0}{4 - 1} = 0.4620981$$

$$b_3 = \frac{\frac{1.791759 - 1.386294}{6 - 4} - 0.4620981}{6 - 1} = -0.051731$$

$$f_2(x) = 0 + 0.4620981(x - 1) - 0.051731(x - 1)(x - 4)$$

$$f_2(x) = 0.5658444$$

which represents a relative error of ε_t = 18.4%. Thus, the curvature introduced by the quadratic formula (Figure 4) improves the interpolation compared with the result obtained using straight lines

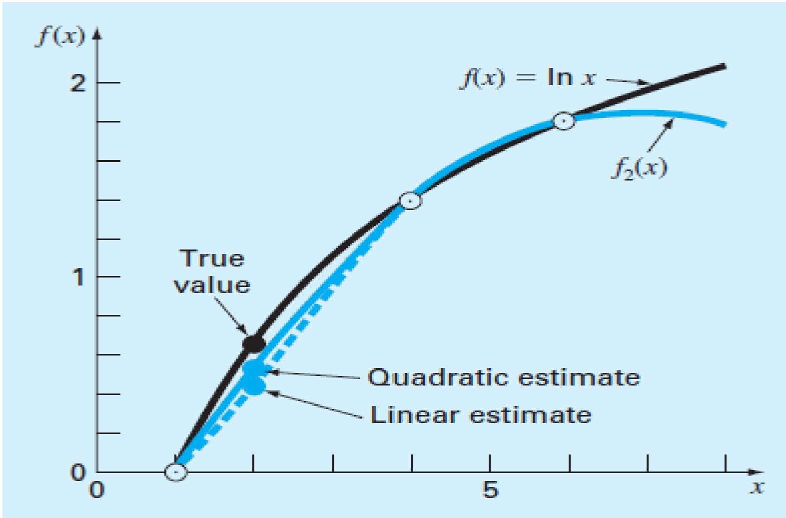


Figure 4: The use of quadratic interpolation to estimate ln 2. The linear interpolation from $x = 1$ to 4 is also included for comparison.

1.4.3 General Form of Newton's Interpolating Polynomials

The preceding analysis can be generalized to fit an $(n - 1)$th-order polynomial to n data points. The $(n - 1)$th-order polynomial is

$$f_{n-1}(x) = b_1 + b_2(x - x_1) + \cdots + b_n(x - x_1)(x - x_2)\cdots(x - x_{n-1})$$

Data points can be used to evaluate the coefficients $b_1, b_2, \ldots, b_n$. For an $(n - 1)$th-order polynomial, n data points are required: $[x_1, f(x_1)], [x_2, f(x_2)], \ldots, [x_n, f(x_n)]$ to evaluate the coefficients:

$$b_1 = f(x_1)$$

$$b_2 = f[x_2, x_1]$$

$$b_3 = f[x_3, x_2, x_1]$$

$$\vdots$$

$$b_n = f[x_n, x_{n-1}, \cdots, x_3, x_2, x_1]$$

where the bracketed function evaluations are finite divided differences. For example, the first finite divided difference is represented generally as

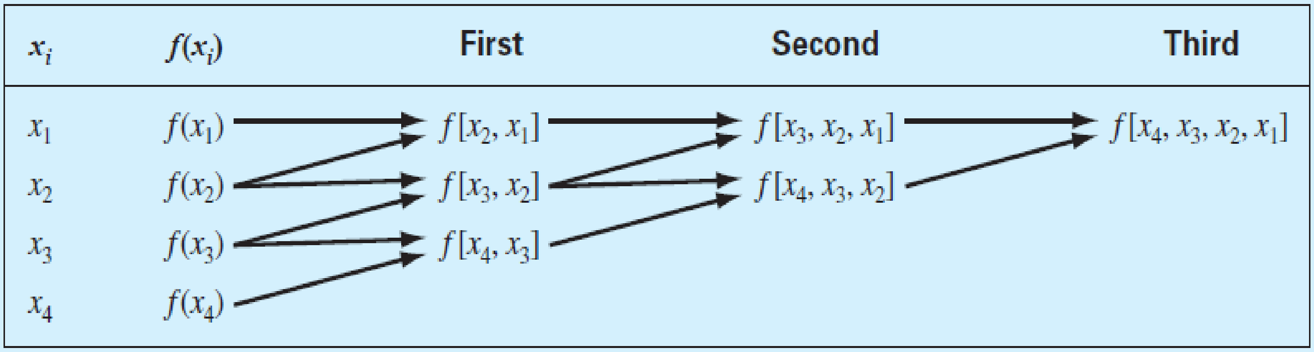
$$f[x_i, x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j}$$

The second finite divided difference, which represents the difference of two first divided differences, is expressed generally as

$$f[x_i, x_j, x_k] = \frac{f[x_i, x_j] - f[x_j, x_k]}{x_i - x_k}$$

Similarly, the *n*th finite divided difference is

$$f[x_n, x_{n-1}, \cdots, x_3, x_2, x_1] = \frac{f[x_n, x_{n-1}, \cdots, x_3, x_2] - f[x_n, x_{n-1}, \cdots, x_3, x_2, x_1]}{x_n - x_1}$$



These differences can be used to evaluate the coefficients, which can then be substituted into the general form of Newton.s interpolating polynomial:

$$f_{n-1}(x) = f(x_1) + (x - x_1)f[x_2, x_1] + (x - x_1)(x - x_2)f[x_3, x_2, x_1] + \cdots$$
$$+ (x - x_1)(x - x_2) \cdots (x - x_{n-1})f[x_n, x_{n-1}, \cdots, x_3, x_2, x_1]$$

Example 3: Employ a third order Newton polynomial to estimate ln 2 with a parabola.

$x_1 = 1$	$f(x_1) = 0$
$x_2 = 4$	$f(x_2) = 1.386294$
$x_3 = 6$	$f(x_3) = 1.791759$
$x_4 = 5$	$f(x_4) = 1.609438$

The third-order polynomial with $n = 4$ is

$$f_3(x) = b_1 + b_2(x - x_1) + b_3(x - x_1)(x - x_2) + b_4(x - x_1)(x - x_2)(x - x_3)$$

The first divided differences

$$f[x_2, x_1] = \frac{1.386294 - 0}{4 - 1} = 0.4620981$$

$$f[x_3, x_2] = \frac{1.791759 - 1.386294}{6 - 4} = 0.2027326$$

$$f[x_4, x_3] = \frac{1.609438 - 1.791759}{5 - 6} = 0.1823216$$

The second divided differences

$$f[x_3, x_2, x_1] = \frac{0.2027326 - 0.4620981}{6 - 1} = -0.05187311$$

$$f[x_4, x_3, x_2] = \frac{0.1823216 - 0.2027326}{5 - 4} = -0.02041100$$

The third divided differences

$$f[x_4, x_3, x_2, , x_1] = \frac{-0.02041100 - (-0.05187311)}{5 - 1} = 0.007865529$$

Thus, the divided difference table is

x_i	$f(x_i)$	First	Second	Third
1	0	0.4620981	−0.05187311	0.007865529
4	1.386294	0.2027326	−0.02041100	
6	1.791759	0.1823216		
5	1.609438			

$$f_3(x) = 0 + 0.4620981(x - 1) - 0.05187311(x - 1)(x - 4)$$
$$+ 0.007865529(x - 1)(x - 4)(x - 6)$$
$$f_3(2) = 0.6287686$$

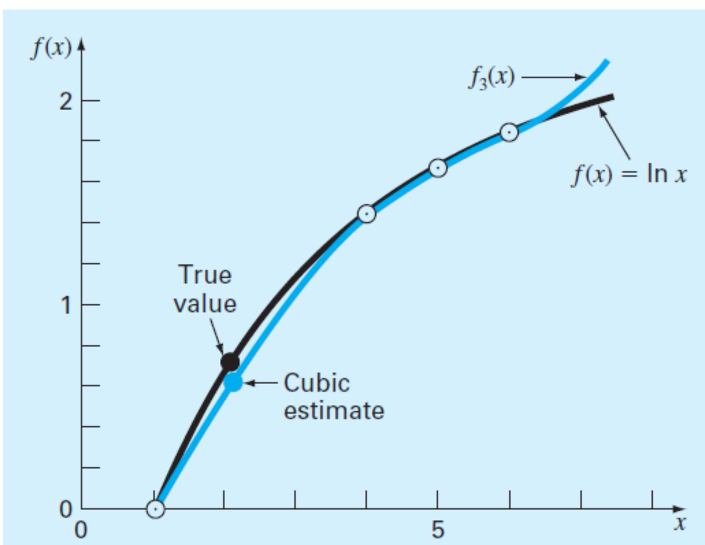


Figure 5: The use of cubic interpolation to estimate ln 2.

Notice about Newton's interpolating**:**

1. General form of Newton's interpolating polynomials is called *Newton's divided-difference interpolating polynomial.*

2. It is not necessary that the data points used be equally spaced.

3. With Newton's interpolation, if you add new points you don't have to re-calculate all the coefficients. For example, the interpolation polynomial in the points x_0, x_1,..., $x_n$with addition the point x_{n+1}then using Newton's method the polynomial in the points x_0, x_1,..., $x_n$can be used to find the polynomial for the points x_0, x_1,..., x_n, x_{n+1}You don't have to find all the coefficients all over again.

1.5 Lagrange Interpolating Polynomial

The Lagrange interpolating polynomial is simply a reformulation of the Newton polynomial that avoids the computation of divided differences. It can be represented as

$$f_{n-1}(x) = \sum_{i=1}^{n} L_i(x) f(x_i)$$

$$L_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

where $\prod$ designates the "product of." For example, the linear version (n = 2) is

$$f_1(x) = L_1 f(x_1) + L_2 f(x_2)$$

$$L_1 = \frac{x - x_2}{x_1 - x_2} \; and \; L_2 = \frac{x - x_1}{x_2 - x_1}$$

Such a second-order Lagrange interpolating polynomial can be written as

$$f_2(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} f(x_1) + \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} f(x_2) + \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)} f(x_3)$$

The drawbacks of Lagrange interpolation is that if we add another support point x_{n+1}, we have to recompute all of the Lagrange polynomials again.

Example 4: Use a Lagrange interpolating polynomial of the first and second order to evaluate the density of unused motor oil at *T* = 15 °C based on the following data:

$x_1 = 0$	$f(x_1) = 3.85$
$x_2 = 20$	$f(x_2) = 0.8$
$x_3 = 40$	$f(x_3) = 0.212$

The first-order polynomial can be used to obtain the estimate at $x = 15$:

$$f_1(x) = \frac{(x - x_2)}{(x_1 - x_2)} f(x_1) + \frac{(x - x_1)}{(x_2 - x_1)} f(x_2)$$

$$f_1(x) = \frac{15 - 20}{0 - 20} 3.85 + \frac{15 - 0}{20 - 0} 0.8 = 1.5625$$

In a similar fashion, the second-order polynomial is developed as

$$f_2(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} f(x_1) + \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} f(x_2) + \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)} f(x_3)$$

$$f_2(x) = \frac{(15 - 20)(15 - 40)}{(0 - 20)(0 - 40)} 3.85 + \frac{(15 - 0)(15 - 40)}{(20 - 0)(20 - 40)} 0.8 + \frac{(15 - 0)(15 - 20)}{(40 - 0)(40 - 20)} 0.212$$
$$= 1.3316875$$

Example 5: Use a Lagrange interpolating polynomial of the first, second and third order to evaluate ln 2

$x_1 = 1$	$f(x_1) = 0$
$x_2 = 4$	$f(x_2) = 1.386294$
$x_3 = 6$	$f(x_3) = 1.791759$
$x_4 = 5$	$f(x_4) = 1.609438$

$$f_1(x) = \frac{(x - x_2)}{(x_1 - x_2)} f(x_1) + \frac{(x - x_1)}{(x_2 - x_1)} f(x_2)$$

$$f_1(x) = \frac{(2 - 4)}{(1 - 4)} (0) + \frac{(2 - 1)}{(4 - 1)} (1.386294) = 0.462098$$

$$f_2(x) = \frac{(2 - 4)(2 - 6)}{(1 - 4)(1 - 6)} (0) + \frac{(2 - 1)(2 - 6)}{(4 - 1)(4 - 6)} (1.386294) + \frac{(2 - 1)(2 - 4)}{(6 - 1)(6 - 4)} (1.791759$$
$$= 0.565844$$

$$f_3(x) = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}f(x_1) + \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}f(x_2)$$
$$+ \frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)}f(x_3) + \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}f(x_4)$$

$$f_3(2) = \frac{(2-4)(2-6)(2-5)}{(1-4)(1-6)(1-5)}\,(0) + \frac{(2-1)(2-6)(2-5)}{(4-1)(4-6)(4-5)}(1.386294)$$
$$+ \frac{(2-1)(2-4)(2-5)}{(6-1)(6-4)(6-5)}(1.791759) + \frac{(2-1)(2-4)(2-6)}{(5-1)(5-4)(5-6)}(1.609438)$$

$$f_3(2) = 0.62876$$

These results agree with those previously obtained using Newton's interpolating polynomial

Exercises:

1. Calculate $f(3.4)$ using the following data using first Newton's interpolating polynomial and second Lagrange polynomial

x	1	2	2.5	3	4	5
$f(x)$	0	5	6.5	7	3	1

2. The acceleration due to gravity at an altitude y above the surface of the earth is given by

y, m	0	30000	60000	90000	120000
$g, m/s^2$	9.81	9.7487	9.6879	9.6278	9.5682

Compute g at y = 55,000 m.

1.6 Polynomial Interpolation with Equally Spaced Base Points

When the base point values are equally spaced so that $x_1 - x_0 = x_2 - x_1 = \cdots = x_n - x_{n-1} = h$, then the interpolating polynomial will be developed in terms of the forward (Δ), backward (∇) and central (δ)difference opertors. Use these operators permits reduction in computation compare with Newton or Lagrange forms

1.6.1 Forward Differences (Newton-Gregory forward)

The forward difference operator is (Δ). $\Delta f(x)$ is termed the first forward difference, $\Delta^2 f(x)$ the second forward difference, etc. the forward difference can be computed and saved in forward difference table in much the same manner as shown earlier for the divided differences

x_i	$f(x_i)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	**−5**				
		6			
1	1		**2**		
		8		**6**	
2	9		8		**0**
		16		6	
3	25		14		
		30			
4	55				

A parameter α is introduced, in order to rewrite the forward finite difference in compact form such that

$$x = x_0 + \alpha h, \quad 0 \le \alpha \le n$$

The forward difference polynomial formula in term ofα is expressed as

$$f(x) = f(x_0) + \alpha\, \Delta f(x_0) + \frac{\alpha(\alpha-1)}{2!}\Delta^2 f(x_0) + \cdots + \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}\Delta^n f(x_0)$$

Example 6: Apply Newton formal to the data and evaluate the second degree polynomial for interpolation argument $x = 1.5$

$$\alpha = \frac{x - x_1}{h} = \frac{1.5 - 0}{1} = 1.5$$

$$f_2(x) = f(x_0) + \alpha\,\Delta f(x_0) + \frac{\alpha(\alpha-1)}{2!}\Delta^2 f(x_0)$$

$$f_2(1.5) = -5 + 6\,(1.5) + \frac{1.5(0.5)}{2!}(2) = 4.75$$

1.6.2 Backward differences (Newton-Gregory backward)

The backward difference operator is (∇). $\nabla f(x)$ is termed the first backward difference, $\nabla^2 f(x)$ the second backward difference, etc.

x_i	$f(x_i)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	-5				
		6			
1	1		2		
		8		6	
2	9		8		**0**
		16		**6**	
3	25		**14**		
		30			
4	**55**				

A parameter α is defined as the origin base point x_n so that

$$x = x_n + \alpha h, \quad -n \le \alpha \le 0$$

$$f(x) = f(x_n) + \alpha\,\nabla f(x_n) + \frac{\alpha(\alpha+1)}{2!}\nabla^2 f(x_n) + \cdots + \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)}{n!}\nabla^n f(x_n)$$

This equation is known as Newton backward formula. Since Newton backward formula uses differences along the lower diagonal of the difference table, it is most useful for interpolation near the end of a set of tabulated values.

Example 7: Find the interpolated value predicted by third degree NBF for $x = 3.5$

$$\alpha = \frac{x - x_n}{h} = \frac{3.5 - 4}{1} = -0.5$$

$$f_3(x) = f(x_n) + \alpha\,\nabla f(x_n) + \frac{\alpha(\alpha+1)}{2!}\nabla^2 f(x_n) + \frac{\alpha(\alpha+1)(\alpha+2)}{3!}\nabla^3 f(x_n)$$

$$f_3(x) = 55 + 30\,\alpha \; + \frac{\alpha(\alpha+1)}{2!}(14) + \frac{\alpha(\alpha+1)(\alpha+2)}{3!}(6)$$

$$f_3(x) = 55 + 30\,\alpha\ + 7\alpha^2 + 7\alpha + \alpha^3 + 3\alpha^2 + 2\alpha$$

$$f_3(x) = 55 + 39\,(x-4)\ + 10(x-4)^2 + (x-4)^3$$

$$f_3(x) = 55 + 39\,x - 156\ + 10x^2 - 80x + 160 + x^3 - 8x^2 + 16x - 4x^2 + 32x - 64$$

$$f_3(x) = x^3 - 2x^2 + 7x - 5$$

$$f_3(3.5) = 55 + (-0.5)(30) + \frac{(0.5)(-0.5)}{2!}(14) + \frac{(-0.5)(0.5)(1.5)}{3!}(6) = 37.875$$

1.6.3 Central Differences (Newton-Gregory Central)

A simple notation which describe differences along a part near the center of the difference table is needed the central difference operator (δ) is defined as follows:

x_i	$f(x_i)$	$\delta f(x)$	$\delta^2 f(x)$	$\delta^3 f(x)$	$\delta^4 f(x)$
	−5				
0		6			
	1		2		
1		8		6	
	9		8		0
2		16		6	
	25		14		
3		30			
	55				
4					

The central difference table (entires are identical with the forward and backward difference tables except for subscripts of the base points) then takes the form central difference table.

If the solid lines is taken, the formulation is termed the Gass forward formula by

$$f(x) = f(x_0) + \alpha\delta f\left(x_0 + \frac{h}{2}\right) + \alpha(\alpha-1)\delta^2\frac{f(x_0)}{2!} + \alpha(\alpha-1)(\alpha-2)\delta^3\frac{f\left(x_0+\frac{h}{2}\right)}{3!}$$

$$+ \alpha(\alpha-1)(\alpha-2)(\alpha-3)\delta^4\frac{f(x_0)}{4!}$$

$$\alpha = \frac{x - x_0}{h}$$

when the dotted lines is used, Gass backward is generated

$$f(x) = f(x_0) + \alpha\delta f\left(x_0 - \frac{h}{2}\right) + \alpha(\alpha+1)\delta^2 \frac{f(x_0)}{2!} + (\alpha-1)\alpha(\alpha+1)\delta^3 \frac{f\left(x_0 - \frac{h}{2}\right)}{3!}$$

Example 8: Use the gauss forward formula and central difference to compute the interpolant value for inpoation argument $x = 2.5$

$$\alpha = \frac{x - x_0}{h} = \frac{2.5 - 2}{1} = 0.5$$

$$f_3(2.5) = 9 + (0.5)(16) + \frac{(0.5)(-0.5)}{2!}(8) + \frac{(0.5)(-0.5)(-1.5)}{3!}(6) = 16.375$$

Exercises 2:

1. Use an appropriate interpolation formula to estimate $f(16.4)$ and $f(23.5)$ from following data

x	16	18	20	22	24
$fx)$	261.3	293.7	330	372.2	422.3

2. Use gauss's forward formula to evaluate y at 30 from the following data

x	21	25	29	33	37
y	18.4708	17.8144	17.1070	16.3432	15.5154