Chapter 4

Initial-Value Problems

4.1 Introduction

This chapter is to solve ordinary differential equations of the form

$$\frac{dy}{dx} = f(x, y)$$

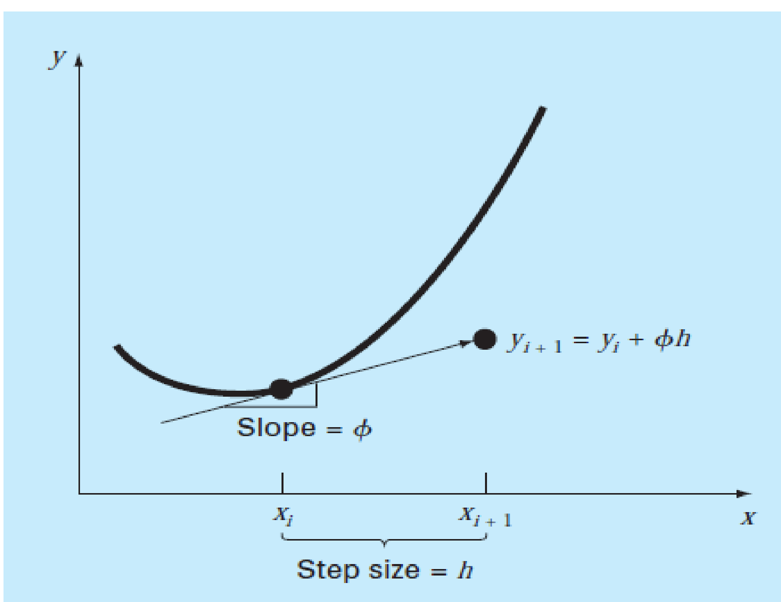
the method was of the general form

$$new\ value = old\ value + slope\ \times step\ size$$

or, in mathematical terms,

$$y_{i+1} = y_i + \varphi h$$

According to this equation, the slope estimate of φ is used to extrapolate from an old value y_i to a new value y_{i+1} over a distance h. The following Figure shows a one-step method. All one-step methods can be expressed in this general form, with the only difference being the manner in which the slope is estimated.



4.2 Euler's Method

The first derivative provides a direct estimate of the slope at x_i as shown in the following Figure:

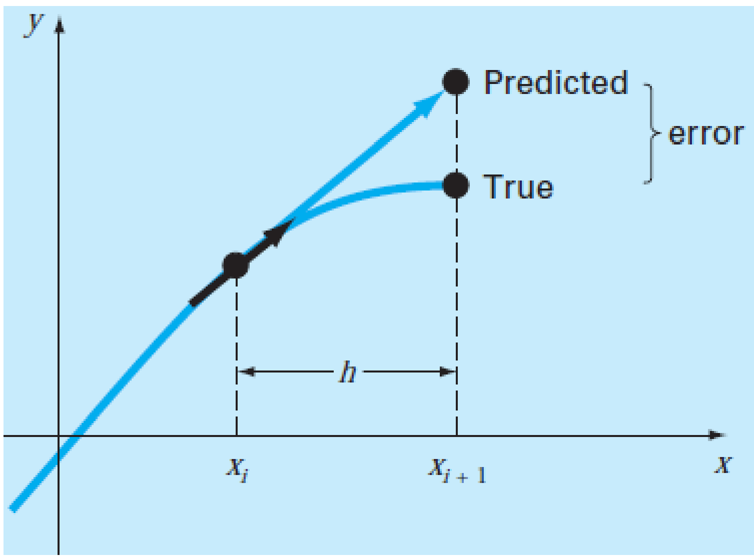
$$\varphi = f(x_i, y_i)$$

where $f(x_i, y_i)$ is the differential equation evaluated at x_i and y_i.

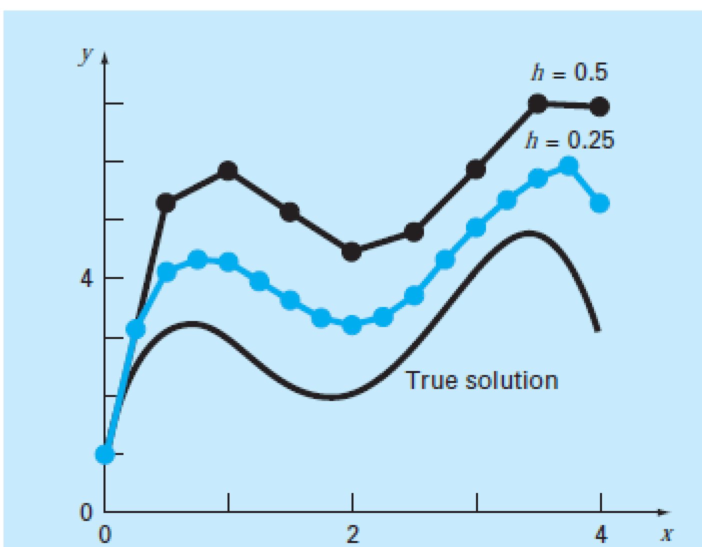
This estimate can be substituted into

$$y_{i+1} = y_i + f(x_i, y_i)h$$

This formula is referred to as *Euler's* (or the *Euler-Cauchy* or the *point-slope*) *method.* A new value of *y* is predicted using the slope (equal to the first derivative at the original value of *x*) to extrapolate linearly over the step size *h*.



Comparison of two numerical solutions with Euler's method using step sizes of 0.5 and 0.25.



Example 1: Use Euler's method to numerically integrate

$$\frac{dy}{dx} = -2\,x^3 + 12\,x^2 - 20\,x + 8.5$$

From $x = 0$ to $x = 4$ with a step size of 0.5. The initial condition at $x = 0$ is $y = 1$. Recall that the exact solution is

$$y = -0.5\,x^4 + 4\,x^3 - 10\,x^2 + 8.5x + 1$$

Solution

One step

$$y(0.5) = y(0) + f(0,1)\,0.5$$

Where $y(0) = 1$ and the slope estimate at $x = 0$ is

$$f(0,1) = -2\,(0)^3 + 12\,(0)^2 - 20\,(0) + 8.5 = 8.5$$

Therefore,

$$y(0.5) = 1 + 8.5\,(0.5) = 5.25$$

The true solution at $x = 0.5$ is

$$y(0.5) = -0.5\,(0.5)^4 + 4\,(0.5)^3 - 10\,(0.5)^2 + 8.5(0.5) + 1 = 3.2175$$

Thus the error

$$E_t = true - approximate = 3.2175 - 5.25 = -2.03125\ or - 63.1\ \%$$

Second step

$$y(1) = y(0.5) + f(0.5, 5.25)\,0.5$$

$$= 5.25 + [-2\,(0.5)^3 + 12\,(0.5)^2 - 20\,(0.5) + 8.5]0.5 = 5.75$$

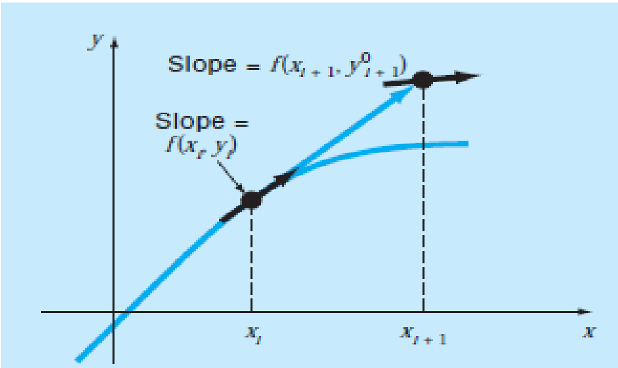
x	y_{true}	y_{Euler}
0	1	1
0.5	3.21875	5.25
1	3	5.75
1.5	2.21875	5.125
2	2	4.5
2.5	2.71875	4.75
3	4	5.75
3.5	4.71875	7.125
4	3	7

4.3 Improvements of Euler's Method

Two simple modifications actually belong to a larger class of solution techniques called Runge-Kutta methods are available:

4.3.1 Heun's Method

One method to improve the estimate of the slope involves the determination of two derivatives for the interval—one at the initial point and another at the end point. The two derivatives are then averaged to obtain an improved estimate of the slope for the entire interval. This approach, called *Heun's method,* is depicted graphically in the following Figure.



(a)

Recall that in Euler's method, the slope at the beginning of an interval

$$y_i' = f(x_i, y_i)$$

is used to extrapolate linearly to y_{i+1}

$$y_{i+1}^0 = y_i + f(x_i, y_i)h$$

The standard Euler method stopped at this point. However, in Heun's method the y_{i+1}^0 is not the final answer, but an intermediate prediction. This is why we have distinguished it with a superscript 0. The equation is called a **predictor equation** Figure (a). It provides an estimate of y_{i+1} that allows the calculation of an estimated slope at the end of the interval:

$$y'_{i+1} = f(x_{i+1}, y_{i+1}^0)$$

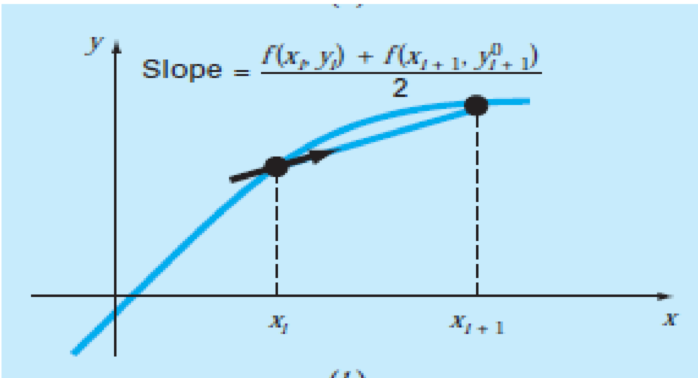
Thus, the two slopes can be combined to obtain an average slope for the interval:

$$\bar{y}' = \frac{y'_i + y'_{i+1}}{2} = \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^0)}{2}$$

This average slope is then used to extrapolate linearly from y_i to y_{i+1}

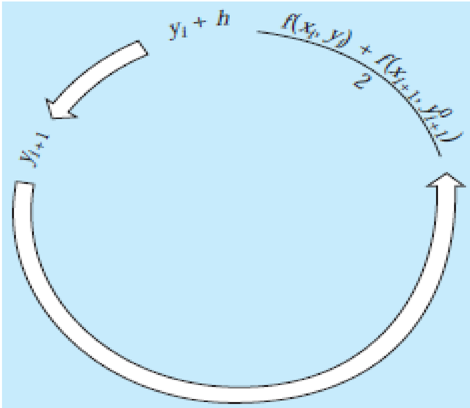
$$y_{i+1} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^0)}{2}h$$

which is called a *corrector equation* Figure (b).



(b)

An old estimate can be used repeatedly to provide an improved estimate of y_{i+1}.



A termination criterion for convergence of the corrector is provided by

$$|\varepsilon_a| = \left|\frac{y_{i+1}^{j} - y_{i+1}^{j-1}}{y_{i+1}^{j}}\right| 100\%$$

Where y_{i+1}^{j-1} and y_{i+1}^{j} are the result from the prior and the present iteration of the corrector, respectively.

Example 2: Use Heun's method to integrate $y' = 4e^{0.8x} - 0.5y$ from $x = 0$ to $x = 4$ with a step size of 1. The initial condition at $x = 0$ is $y = 2$

Note that the following analytical solution is used to generate the true solution values

$$y = \frac{4}{1.3}(e^{0.8x} - e^{-0.5x}) + 2e^{-0.5x}$$

Solution: First, the slope at initial condition

$$y_0' = 4e^{0} - 0.5(2) = 3$$

By using predictor to obtain an estimate of y at 1:

$$y_1^0 = 2 + 3(1) = 5$$

To predict the slope at the end of the interval

$$y_1' = f(x_1, y_1^0) = 4e^{0.8(1)} - 0.5(5) = 6.40214$$

which can be combined with the initial slope to yield an average slope over the interval from *x* = 0 to 1

$$y' = \frac{3 + 6.402164}{2} = 4.701082$$

which is closer to the true average slope of 4.1946. This result can then be substituted into the corrector to give the prediction at *x* = 1

$$y_1 = 2 + 4.701082(1) = 6.701082$$

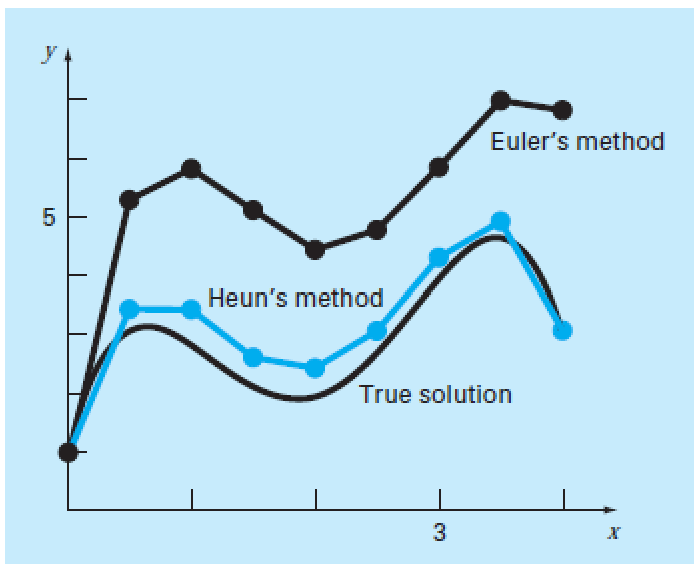
which represents a true percent relative error of -8.18%. Thus, the Heun method without iteration of the corrector reduces the absolute value of the error by a factor of about 2.4 as compared with Euler's method. At this point, we can also compute an approximate error as

$$y_1 = 2 + \frac{[3 + 4e^{0.8(1)} - 0.5(6.701082)]}{2}(1) = 6.382129$$

which represents an $|\varepsilon t|$ of 3.03%.0

x	y_{true}	Iterations of Heun's Method			
		1		15	
		y_{Heun}	$\lvert\varepsilon_t\rvert\%$	y_{Heun}	$\lvert\varepsilon_t\rvert\%$
0	2	2	0	2	0
1	6.1946314	6.7010819	8.18	6.3608655	2.68
2	14.8439219	16.3197819	9.94	15.3022367	3.09
3	33.6771718	37.1992489	10.46	34.7432761	3.17
4	75.3389626	83.3377674	10.62	77.7350962	3.18

The following Figure shows comparison of the true solution with a numerical solution using Euler's and Heun's methods



4.3.2 The Midpoint (or modified Euler) Method

This technique uses Euler's method to predict a value of y at the midpoint of the interval (Figure *a*):

$$y_{i+1/2} = y_i + f(x_i, y_i)\frac{h}{2}$$

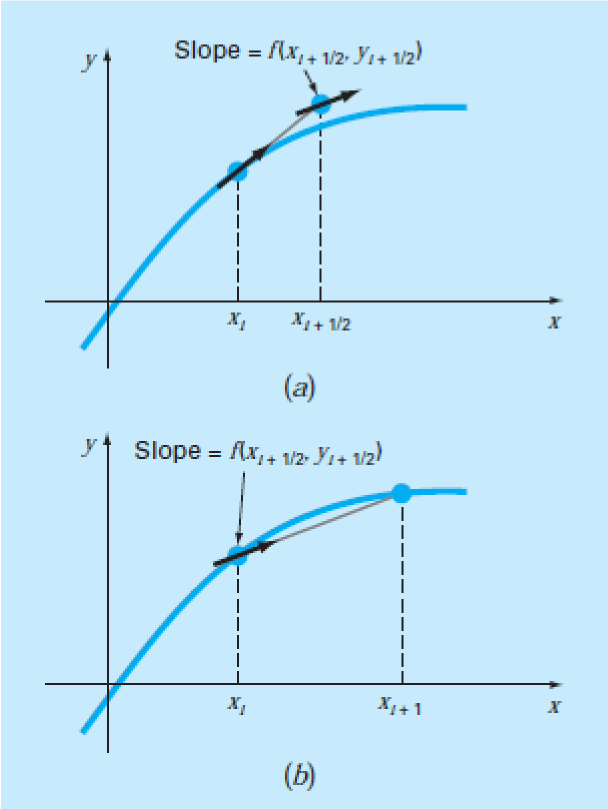
This predicted value is used to calculate a slope at the midpoint:

$$y'_{i+1/2} = f\left(x_{i+1/2}, y_{i+1/2}\right)$$

which is assumed to represent a valid approximation of the average slope for the entire interval.

This slope is then used to extrapolate linearly from x_i to x_{i+1}

$$y_{i+1} = y_i + f\left(x_{i+1/2}, y_{i+1/2}\right)h$$



4.4 Runge-Kutta Methods

Runge-Kutta (*RK*) methods can be cast in the generalized form

$$y_{i+1} = y_i + \emptyset(x_i, y_i, h)h$$

$$\emptyset = a_1k_1 + a_2k_2 + \cdots + a_nk_n$$

where $\emptyset(x_i, y_i, h)$ is called an *increment function*, which can be interpreted as a representative slope over the interval. The increment function can be written in general form as where the a's are constants and the k's are

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1h, y_i + q_{11}k_1h)$$

$$k_3 = f(x_i + p_2h, y_i + q_{21}k_1h + q_{22}k_2h)$$

$$\vdots$$

$$k_n = f\left(x_i + p_{n-1}h, y_i + q_{n-1,1}k_1h + q_{n-1,2}k_2h + \cdots + q_{n-1,n-1}k_{n-1}h\right)$$

where the p's and q's are constants.

Notice that the first-order RK method with $n = 1$ is Euler's method.

4.5 Second-Order Runge-Kutta Methods

The second-order version is

$$y_{i+1} = y_i + (a_1k_1 + a_2k_2)h$$

Where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1h, y_i + q_{11}k_1h)$$

values for a_1, a_2, p_1, and q_{11} are evaluated by the three equations are

$$a_1 + a_2 = 1$$

$$a_2p_1 = \frac{1}{2}$$

$$a_2q_{11} = \frac{1}{2}$$

Suppose that a value for a_2 is specified then can be solved simultaneously for

$$a_1 = 1 - a_2$$

$$p_1 = q_{11} = \frac{1}{2a_2}$$

Because we can choose an infinite number of values for a_2, there are an infinite number of second order RK methods.

4.5.1 Heun Method with a Single Corrector ($a_2 = \frac{1}{2}$)

If a_2 is assumed to be 1/2, can be solved for $a_1 = \frac{1}{2}$ and $p_1 = q_{11} = 1$.

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

Where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + h)$$

Note that k_1 is the slope at the beginning of the interval and k_2 is the slope at the end of the interval. Consequently, this second-order Runge-Kutta method is actually Heun's technique without iteration.

4.5.2 The Midpoint Method (a_2 = 1).

If a_2 is assumed to be 1, then a_1 = 0, $p_1 = q_{11} = \frac{1}{2}$

$$y_{i+1} = y_i + k_2 h$$

Where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

This is the midpoint method.

4.3.3 Ralston's Method ($a_2 = \frac{2}{3}$).

By choosing $a_2 = \frac{2}{3}$provides a minimum bound on the truncation error for the second-order RK algorithms. For this version, $a_1 = \frac{1}{3}$and$p_1 = q_{11} = \frac{3}{4}$ and yields

$$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$$

Where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$$

Example 3: Use the midpoint method and Ralston's method to numerically integrate

$$f(x, y) = -2\,x^3 + 12\,x^2 - 20\,x + 8.5$$

from $x = 0$ to $x = 4$ using a step size of 0.5. The initial condition at $x = 0$ is $y = 1$.

Solution: The first step in the midpoint method is to compute

$$k_1 = -2\,(0)^3 + 12\,(0)^2 - 20\,(0) + 8.5 = 8.5$$

However, because the ODE is a function of x only, so

$$k_2 = -2\,(0.25)^3 + 12\,(0.25)^2 - 20\,(0.25) + 8.5 = 4.21875$$

The slope at the midpoint is

$$y(0.5) = 1 + 4.21875(0.5) = 3.109375\ \varepsilon_t = 3.4\%$$

For Ralston's method,

$$k_1 = -2\,(0)^3 + 12\,(0)^2 - 20\,(0) + 8.5 = 8.5$$

$$k_2 = -2\,(0.375)^3 + 12\,(0.375)^2 - 20\,(0.375) + 8.5 = 2.58203125$$

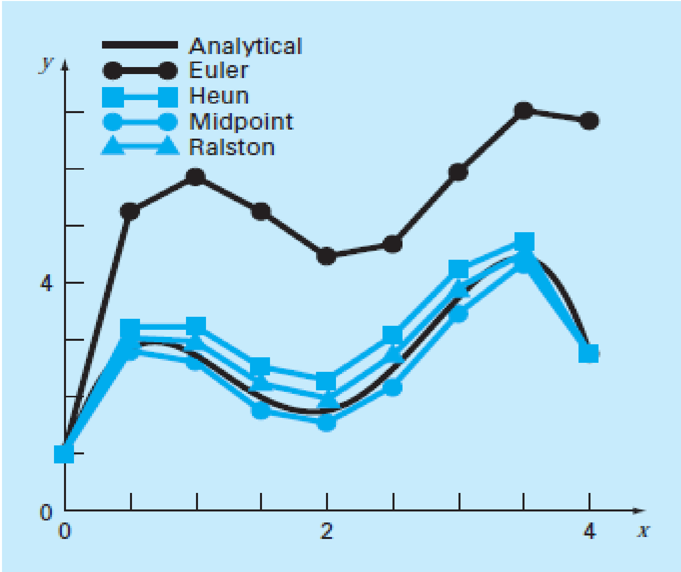
The average slope is computed by

$$\emptyset = \frac{1}{3}(8.5) + \frac{2}{3}(2.58203125) = 4.5546875$$

which can be used to predict

$$y(0.5) = 1 + 4.5546875(0.5) = 3.27734375\ \varepsilon_t = -1.82\%$$

x	y_{true}	y_{Heun}	$\lvert\varepsilon_t\rvert\%$	$y_{midpoint}$	$\lvert\varepsilon_t\rvert\%$	$y_{Ralston}$	$\lvert\varepsilon_t\rvert\%$
0	1	1	0	1	0	1	0
0.5	3.21875	3.4375	6.8	3.109375	3.4	3.277344	1.8
1	3	3.375	12.5	2.8125	6.3	3.101563	3.4
1.5	2.21875	2.6875	21.1	1.984375	10.6	2.347656	5.8
2	2	2.5	25	1.75	12.5	2.140625	7
2.5	2.71875	3.1875	7.2	2.484375	8.6	2.855469	5
3	4	4.375	9.4	3.8125	4.7	4.117188	2.9
3.5	4.71875	4.9375	4.6	4.609375	2.3	4.800781	1.7
4	3	3	0	3	0	3.03125	1



4.6 Third-Order Runge-Kutta Methods

For $n = 3$,

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 4k_2 + k_3)h$$

Where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f(x_i + h, y_i - k_1 h + 2k_2 h)$$

4.7 Fourth-Order Runge-Kutta Methods

The most popular RK methods are fourth order.

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

Where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f(x_i + h, y_i + k_3 h)$$

Example 4: Use the fourth order RK method to numerically integrate

$$(a)\ f(x, y) = -2\,x^3 + 12\,x^2 - 20\,x + 8.5$$

from $x = 0$ to $x = 4$ using a step size of 0.5. The initial condition at $x = 0$ is $y = 1$.

(b) $y' = 4e^{0.8x} - 0.5y$ from $x = 0$ to $x = 4$ with a step size of 1. The initial condition at $x = 0$ is $y = 2$

Solution: (a)

$$k_1 = 8.5, k_2 = 4.21875, k_3 = 4.21875\ and\ k_4 = 1.25$$

$$y(0.5) = 1 + \frac{1}{6}(8.5 + 2(4.21875) + 2(4.21875) + 1.25)0.5 = 3.21875$$

Solution: (b)

$$k_1 = f(0, 2) = 4e^{0.8(0)} - 0.5(2) = 3$$

$$y(0.25) = 2 + 3(0.25) = 2.75$$

$$k_2 = f(0.25, 2.75) = 4e^{0.8(0.25)} - 0.5(2.75) = 3.510611$$

$$y(0.25) = 2 + 3.510611(0.25) = 2.877653$$

$$k_3 = f(0.25, 2.877653) = 4e^{0.8(0.25)} - 0.5(2.877653) = 3.446785$$

$$y(0.5) = 2 + 3.071785(0.5) = 3.723392$$

$$k_4 = f(0.5, 3.723392) = 4e^{0.8(0.5)} - 0.5(3.723392) = 4.105603$$

$$\emptyset = \frac{1}{6}(3 + 2(3.510611) + 2(3.446785) + 4.105603) = 3.503399$$

$$y(0.5) = 2 + 3.503399(0.5) = 3.751699$$