Chapter 2

First Order Differential Equations

2.1 Solution Curves without a Solution

2.1.1 Direction Fields

A derivative $\frac{dy}{dx}$ of a differentiable function gives slopes of tangent lines at points on its graph.

- **Slope**

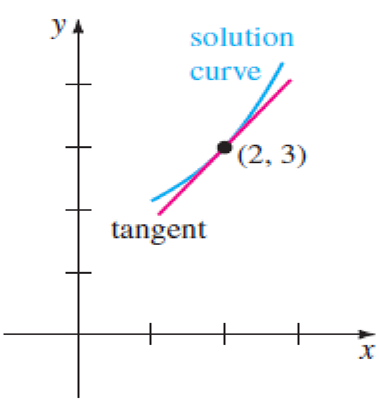
The function f in the normal form $\frac{dy}{dx} = f(x, y)$ is called the slope function or rate function. The slope of the tangent line at $(x, y(x))$on a solution curve is the value of the first derivative$\frac{dy}{dx}$ at this point.

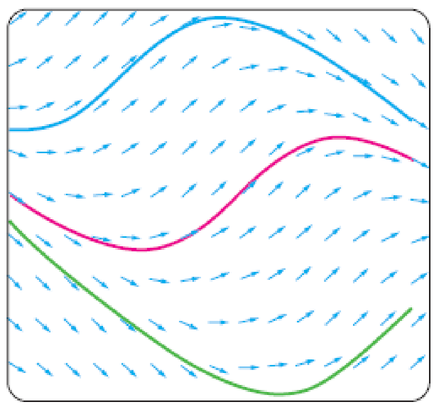
For example, consider the equation $\frac{dy}{dx} = 0.2xy$, where $f(x, y) = 0.2xy$. At the point (2, 3) the slope of a lineal element is $f(2, 3) = 0.2(2)(3)$=1.2.

- **Direction Field**

Direction field or Slope field of the differential equation $\frac{dy}{dx} = f(x, y)$ is to evaluate f over a rectangular grid of points in the *xy*-plane and draw a line element at each point (*x*, *y*) of the grid with slope f (*x*, *y*), then the collection of all these line elements.

Visually, the direction field suggests the appearance or shape of a family of solution curves of the differential equation. The following Figure shows a computer-generated direction field of the differential equation $\frac{dy}{dx} = \sin(x + y)$over a region of the xy-plane. A single solution curve that passes through a direction field must follow the flow pattern of the field; it is tangent to a lineal element when it intersects a point in the grid.





- **Increasing/Decreasing**

If $\frac{dy}{dx} > 0$ (or $\frac{dy}{dx} < 0$) for all *x* in an interval *I*, then a differentiable function *y* =*y*(*x*) is increasing (or decreasing) on *I*.

2.1.2 Autonomous First-Order DEs

An ordinary differential equation in which the independent variable does not appear explicitly is said to be **autonomous.** If the symbol *x* denotes the independent variable, then an autonomous first-order differential equation can be written as $f(y, y') = 0$ or in normal form as $\frac{dy}{dx} = f(y)$

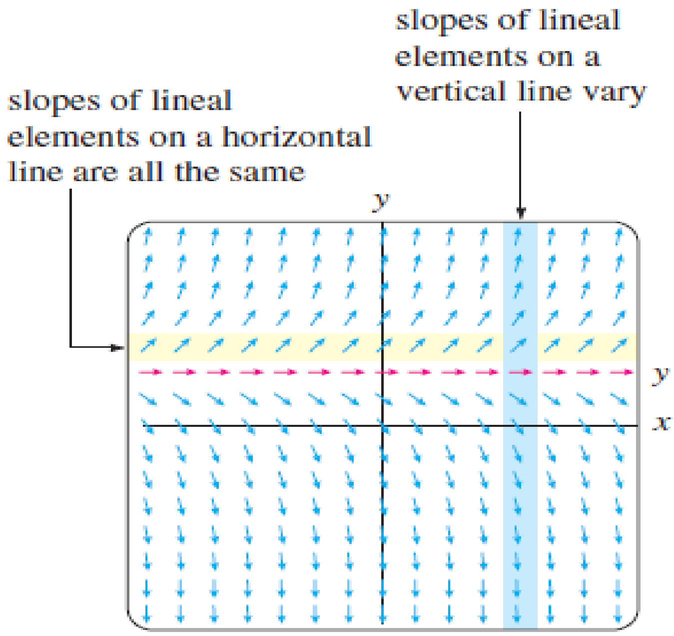
The first-order equations are autonomous and nonautonomous, respectively.

$$\frac{dy}{dx} = \overbrace{1 + y^2}^{f(y} \qquad and\ \frac{dy}{dx} = \overbrace{0.2xy}^{f(x,y)}$$

- **Autonomous DEs and Direction Fields**

If a first-order differential equation is autonomous, then slopes of lineal elements through points in the rectangular grid used to construct a direction field for the DE depend solely on the *y*-coordinate of the points.

Lineal elements passing through points on any *horizontal* line must all have the same slope and therefore are parallel; slopes of lineal elements along any *vertical* line will vary.



Direction field for $\frac{dy}{dx} = 2(y-1)$

2.2 Separable Equations

➢ First Order Separable ODE

A first order ODE of the form

$$\frac{dy}{dx} = f(x)g(y) \quad or \quad g(y)dy = f(x)dx$$

is said to be separable or to have separable variable.

For example, $\frac{dy}{dx} = y^2xe^{3x+4y}$ is separable since $\frac{dy}{dx} = (y^2e^{4y})(xe^{3x})$

$\frac{dy}{dx} = y + sinx$ is not separable

There is no need to use two constants in the integration of a separable equation, because if we write $H(y) + c_1 = G(x) + c_2$, then the difference $c_2 - c_1$ can be replaced by a single constant c. Thus, multiples of constants or combinations of constants can sometimes be replaced by a single constant.

Examples: Which of the following differential equations are separable?

(1) $\frac{dy}{dx} = \frac{x-5}{y^2}$ $\qquad \frac{dy}{dx} = (x-5)(\frac{1}{y^2})$

(2) $\frac{dy}{dx} = \frac{y-1}{x+3}$ $\qquad \frac{dy}{dx} = (\frac{1}{x+3})(y-1)$

(3) $y' = xy - ye^x$ $\qquad y' = y(x - e^x)$

(4) $y' - 2xy = x^2$ $\qquad y' = 2xy + x^2 = x(2y + x)$ $\qquad$ *Not separable*

(5) $y' = -2y + e^x$ $\qquad$ *NO*

(6) $y' = xy - ye^x$ $\qquad = y(x - e^x)$

(7) $y' - 2xy = x$ $\qquad y' = x(2y + 1)$

➢ How to Solve a Separable Differential Equation

1. Separate: $\frac{1}{g(y)}dy = f(x)dx$
2. Integrate (if you can) $\int \frac{1}{g(y)}dy = \int f(x)dx$
3. Solve for y (if possible)

Examples: Solve

$$(1+x)dy - ydx = 0$$

$$\int \frac{dy}{y} = \int \frac{dx}{1+x}$$

$$\ln|y| = \ln|1+x| + c_1$$

$$y = e^{\ln|1+x|+c_1} = e^{\ln|1+x|}.e^{c_1}$$

$$= |1+x|e^{c_1}$$

$$= \pm e^{c_1}(1+x) = c(1+x)$$

$$(3)\ \frac{dy}{dx} = \frac{x-5}{y^2}$$

$$\frac{dy}{dx} = (x-5)(\frac{1}{y^2})$$

$$\int y^2 dy = \int (x-5)dx$$

$$\frac{1}{3}y^3 = \frac{1}{2}x^2 - 5x + C$$

$$y^3 = \frac{3}{2}x^2 - 15x + C$$

$$y = \sqrt[3]{\frac{3}{2}x^2 - 15x + C}$$

$$y' = xy - ye^x$$

$$\int \frac{1}{y}dy = \int (x - e^x)dx$$

$$lny = \frac{1}{2}x^2 - e^x + C$$

$$y = e^{\frac{1}{2}x^2 - e^x + C} = e^{\frac{1}{2}x^2 - e^x}.e^C = C_1 e^{\frac{1}{2}x^2 - e^x}$$

$$y' - 2xy = x$$

$$y' = 2xy + x$$

$$\frac{dy}{dx} = x(2y+1)$$

$$\int \frac{1}{2y+1} dy = \int x dx$$

$$\frac{1}{2}\ln(2y+1) = \frac{1}{2}x^2 + C$$

$$\ln(2y+1) = x^2 + C_1$$

$$(2y+1) = e^{x^2+C_1}$$

$$(2y+1) = k_0 e^{x^2}$$

$$y = ke^{x^2} - \frac{1}{2}$$

Example: Solve IVP

$$y' = 3y, \;\; y(0) = -2$$

$$\frac{dy}{dx} = 3y$$

$$\int \frac{1}{y} dy = \int 3dx$$

$$ln|y| = 3x + C$$

$$|y| = e^{3x+C}$$

$$y = C_0 e^{3x}$$

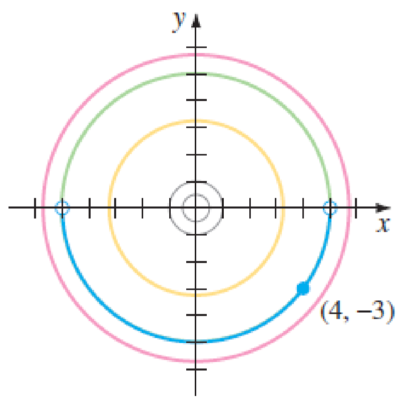
$$y(0) = C_0 e^0 = -2$$

$$C_0 = -2$$

$$y = -2\, e^{3x}$$

Example: Solve IVP

$$\frac{dy}{dx} = -\frac{x}{y}\,, y(4) = -3$$



$$\int y\, dy = -\int x\, dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + C_0$$

$$x^2 + y^2 = c^2$$

$$at\ x = 4\ and\ y = -3$$

$$x^2 + y^2 = 25$$

$$y = \pm\sqrt{25 - x^2}$$

The two functions $y = \emptyset_1(x) = +\sqrt{25 - x^2}\ and\ y = \emptyset_2(x) = -\sqrt{25 - x^2}$

The solution is

$$y = \emptyset_2(x) = -\sqrt{25 - x^2}$$

Example: Solve

$$\frac{dy}{dx} = y^2 - 4$$

$$\int \frac{dy}{y^2 - 4} = \int dx \Rightarrow \int \left[\frac{\frac{1}{4}}{y-2} - \frac{\frac{1}{4}}{y+2}\right] dy = \int dx$$

$$\frac{1}{4}\ln|y-2| - \frac{1}{4}\ln|y+2| = x + c_1$$

$$\frac{1}{4}\ln\left|\frac{y-2}{y+2}\right| = 4x + c_2 \Rightarrow \frac{y-2}{y+2} = \pm e^{4x+c_2}$$

$$y = 2\,\frac{1 + ce^{4x}}{1 + ce^{4x}}$$

H.W: Solve

$$(e^{2y} - y)\cos x\,\frac{dy}{dx} = e^{y}\sin 2x, \quad y(0) = 0$$

Exercises

1. Solve the given differential equation by separation of variables.

(a) $\frac{dy}{dx} = \sin 5x$

(b) $dx + e^{3x}dy = 0$

$(c) x\dfrac{dy}{dx} = 4y$

$(d) \dfrac{dy}{dx} = e^{3x+2y}$

(e)$\csc y\ dx + sec^{\,2} x\ dy = 0$

(f) $\sin\ 3x\ dx + 2y\ cos^3 3x dy = 0$

$(g)\ x(1 + y^2)^{1/2}dx = y(1 + x^2)^{1/2}dy$

2. Find an explicit solution of the given initial-value problem.

$$(a)\frac{dx}{dt} = 4(x^2+1), x\left(\frac{1}{4}\right) = 1$$

$$(b)\; x^2\frac{dx}{dt} = y - xy, y(-1) = -1$$

2.3 Linear Equations

A first order differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Is said to be a linear equation

When g(x) =0, the linear equation is said to be homogeneous otherwise is nonhomogeneous.

The standard form of of a linear equation

$$\frac{dy}{dx} + P(x)y = f(x)$$

A solution of this linear DE is on interval I for which both coefficient functions P and f are continuous.

Example: Put the DE in standard form

$$y' - 2xy = x^2$$

$$\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = xcosx \qquad \rightarrow \qquad \frac{dy}{dx} - \frac{2}{x}y = x^2cosx$$

- **Solving a linear first order equation**

1. Remember to put a linear equation into the standard form
2. Identify P(x) and find the integrating factor

$$u(x) = e^{\int p(x)dx}$$

Notice: no constant need be used in evaluing the indefinite integral $\int p(x)\,dx$

3. Multiply the both sides of the standard form equation by the integrating factor.

$$u(x)y' + u(x)p(x)y = f(x)u(x)$$

4. Integrate both sides of the last equation and solve for y

$$\frac{d}{dx}(u(x).y) = u(x)f(x)$$

$$y = e^{-\int p(x)dx} \int e^{\int p(x)dx} f(x)dx + ce^{-\int p(x)dx}$$

Example: Solve

$$\frac{dy}{dx} - 3y = 0 \qquad \frac{dy}{dx} + P(x)y = f(x)$$

SOLUTION: This linear equation can be solved by separation of variables. Alternatively, since the differential equation is already in standard form,

- Identify $P(x) = -3\ and\ f(x) = 0$
- Integrating factor is $e^{\int p(x) - 3dx} = e^{-3x}$.
- Multiply the given equation by this factor

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = e^{-3x}.0\ \ is\ the\ same\ as\ \frac{d}{dx}[e^{-3x}y] = 0$$

- Integration of the last equation

$$\int \frac{d}{dx}(e^{-3x}y) = \int 0\,dx \quad \rightarrow \quad e^{-3x}y = C \quad or\ y = Ce^{3x} \quad on -\infty < x < \infty$$

Example: Solve

$$\frac{dy}{dx} - 3y = 6 \qquad \frac{dy}{dx} + P(x)y = f(x)$$

SOLUTION: This linear equation can be solved by separation of variables. Alternatively, since the differential equation is already in standard form,

- Identify $P(x) = -3\ and\ f(x) = 6$
- Integrating factor is $e^{\int p(x) - 3dx} = e^{-3x}$.
- Multiply the given equation by this factor

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = e^{-3x}.6\ \ is\ the\ same\ as\ \frac{d}{dx}[e^{-3x}y] = 6\,e^{-3x}$$

- Integration of the last equation

$$\int \frac{d}{dx}(e^{-3x}y) = \int 6\,e^{-3x}dx \quad \rightarrow \quad e^{-3x}y = -6(\frac{e^{-3x}}{3} + C \quad or\ y = -2 + Ce^{3x} \quad on -\infty < x < \infty$$

Example: Solve

$$x\,\frac{dy}{dx} - 4y = x^6 e^x$$

- Dividing by x, the standard form is

$$\frac{dy}{dx} - \frac{4}{x} y = x^5 e^x$$

- Identify $P(x) = \frac{4}{x}$ $and\ f(x) = x^5 e^x\ are\ continuous\ on\ (0, \infty)$
- Integrating factor is $e^{\int -4dx/x} = e^{-4\ln x} = e^{\ln x^{-4}} = x^{-4}$
- Multiply by x^{-4} and rewrite

$$x^{-4}\frac{dy}{dx} - 4x^{-5}y = xe^x\ as\ \frac{d}{dx}[x^{-4}y] = x\,e^x$$

- Integration of the last equation

$$\int \frac{d}{dx}[x^{-4}y] = \int x\,e^x\,dx \quad \rightarrow \quad x^{-4}y = x\,e^x - e^x + c\ or\ y = x^5\,e^x - x^4 e^x + cx^4 \quad on\ (0, \infty)$$

Example: Find the general solution of

$$(x^2 - 9)\frac{dy}{dx} + xy = 0$$

- Dividing by $(x^2 - 9)$, the standard form is

$$\frac{dy}{dx} + \frac{x}{(x^2 - 9)} y = 0$$

- Identify

$$P(x) = \frac{x}{(x^2-9)} \quad and\ f(x) = 0\ are\ continuous\ on\ (-\infty, 0), (-3, 3) and (3, \infty)$$

- Integrating factor is $e^{\int x dx/(x^2-9)} = e^{\frac{1}{2}\ln|x^2-9|} = \sqrt{x^2 - 9}$
- Multiply by $\sqrt{x^2 - 9}$ and rewrite

$$\sqrt{x^2 - 9}\frac{dy}{dx} - \frac{x\sqrt{x^2 - 9}}{(x^2 - 9)} y = 0\ as\ \frac{d}{dx}\left[\sqrt{x^2 - 9}\,y\right] = 0$$

- Integration of the last equation

$$\int \frac{d}{dx}\left[\sqrt{x^2 - 9}\,y\right] = \int 0\,dx$$

$$\sqrt{x^2 - 9}\,y = c\ \ or\ y = \frac{c}{\sqrt{x^2 - 9}} \quad on\ (-\infty, 0)\ and (3, \infty)$$

Example: An initial value problem

Solve

$$\frac{dy}{dx} + y = x, \quad y(0) = 4$$

- Identify

  $P(x) = 1 \ \ and\ f(x) = x\ are\ continuous\ on\ (-\infty, \infty)$

- Integrating factor is $e^{\int dx} = e^x$
- Multiply by $e^x$and rewrite

$$e^x \frac{dy}{dx} + e^x y = xe^x\ as\ \frac{d}{dx}[e^x\, y] = xe^x$$

- Integration of the last equation

$$\int \frac{d}{dx}[e^x y] = \int xe^x dx$$

$$e^x\, y = xe^x - e^x + c\ \ or\ y = x - 1 + ce^{-x}$$

- Substituting $y = 4\ at\ x = 0$

$$y = x - 1 + 5e^{-x}\ \ on\ (-\infty, \infty)$$

Example: Solve

$$y' - 2xy = x$$

$$p(x) = -2x \quad \rightarrow \quad u(x) = e^{\int -2x dx} = e^{-x^2}$$

$$e^{-x^2} y' - 2xe^{-x^2} y = e^{-x^2}.x$$

$$\int \frac{d}{dx}\left(e^{-x^2} y\right) = \int e^{-x^2} x dx \quad \rightarrow \quad (e^{-x^2} y = -\frac{1}{2} e^{-x^2} + C)$$

$$e^{x^2}(e^{-x^2} y = -\frac{1}{2} e^{-x^2} + C)$$

$$y = -\frac{1}{2} + Ce^{x^2}$$

Example: Solve

$$\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x cos x, \quad x > 0$$

$$\frac{dy}{dx} - \frac{2y}{x} = x^2 cosx \qquad p(x) = -\frac{2}{x} \quad , \qquad u(x) = e^{\int -\frac{2}{x}dx} = e^{(-2)lnx}$$

$$u(x) = x^{-2}$$

$$\frac{d}{dx}(x^{-2}y) = cosx$$

$$x^{-2}y = \int cosx\, dx = sinx + C$$

$$y = x^2 sinx + Cx^2$$

Example: Solve

$$xy' + 3y = \frac{lnx}{x}$$

$$\frac{dy}{dx} + \frac{3}{x}y = \frac{lnx}{x^2} \qquad p(x) = \frac{3}{x} \qquad u(x) = e^{\int \frac{3}{x}dx} = e^{3lnx} = x^3$$

$$\int \frac{d}{dx}(x^3.y) = x^3 y \quad , \qquad \frac{lnx}{x^2} = \int x\, lnx\, dx$$

$$x^3 y = \frac{1}{2}x^2 lnx - \int \frac{1}{2}x\, dx \qquad u = lnx \quad dv = xdx \quad , du = \frac{1}{x}dx \quad v = \frac{1}{2}x^2$$

$$x^3 y = \frac{1}{2}x^2 lnx - \frac{1}{4}x^2 + C$$

$$y = \frac{lnx}{2x} - \frac{1}{4x} + \frac{C}{x^3}$$

Example: Solve IVP

$$\mathrm{y}' + 2\mathrm{y} = 4, \quad \mathrm{y}(0) = 5$$

$$\mathrm{p(x)} = 2, \quad \mathrm{u(x)} = e^{\int 2dx} = e^{2x}$$

$$e^{2x}y = \int 4e^{2x}\, dx$$

$$e^{2x}y = 2e^{2x} + C$$

$$\mathrm{y} = 2 + \mathrm{C}e^{2x} \quad \rightarrow \quad y(0) = 2 + Ce^0 = 5 \quad \rightarrow \quad C = 3$$

Exercises

1. Find the general solution of the given differential equation.

$$(a)\frac{dy}{dx} = 5y$$

$$(b)\frac{dy}{dx} + y = e^{3x}$$

$$(c) y' + 3x^2y = x^2$$

$$(d)\ x^2y' + xy = 1$$

2. Solve the given initial-value problem. Give the largest interval *I* over which the solution is defined.

$$(a)\frac{dy}{dx} = x + 5y, y(0) = 3$$

$$(b) x\,y' + y = e^x, y(1) = 2$$

2.4 Exact Equations

If $z = f(x, y)$ is a function of two variable with continuous first partial derivativtives in a region of *xy* plane, then its differential is

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

In the special case when $f(x, y) = c$, where c is a constant then

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

For example, if $x^2 - 5xy + y^3 = c$, then the first order DE

$$(2x - 5y)dx + (-5x + 3y^2)dy = 0$$

A differential expression $M(x, y)\,dx + N(x, y)\,dy$ is an **exact differential** in a region *R* of the *xy* plane. A first-order differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be an **exact equation** if the expression on the left-hand side is an exact differential. For example $x^2y^3dx + x^3y^2dy = 0$ is an exact equation because its left hand side is an exact differential

$$d\left(\frac{1}{3}x^3y^3\right) = x^2y^3dx + x^3y^2dy$$

A necessary and sufficient condition that DE be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- **Method of solution given an equation in the differential form *M*(*x*, *y*) *dx* + *N*(*x*, *y*) *dy=0***
    - Determine *M(x, y) and N(x, y)*
    - *If*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

    - Exist a function *f*

$$f(x, y) = \int M(x, y)dx + g(y)$$

    - Identify $g'(y)$

$$g'(y) = N(x, y) - \frac{\partial}{\partial y}\int M(x, y)dx$$

    - Integrate $g'(y)$ and substitute in $f(x, y) = \int M(x, y)dx + g(y)$
    - *f(x, y)= c*

In case using $N(x, y)$

- $f(x, y) = \int N(x, y)dy + h(x)$
- $h'(x) = M(x, y) - \frac{\partial}{\partial x}\int N(x, y)dy$
- Integrate $h'(x)$ and substitute in $f(x, y) = \int N(x, y)dy + h(x)$
- *f(x, y)= c*

Example: Solve

$$2xydx + (x^2 - 1)dy = 0$$

$$M(x, y) = 2xy \; and \; N(x, y) = x^2 - 1$$

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$$

$$M(x,y) = 2xy \; and \; N(x,y) = x^2 - 1$$

$$f(x,y) = \int 2xydx + g(y) = x^2y + g(y)$$

$$g'(y) = (x^2 - 1) - x^2 \Longrightarrow \int g'(y)\,dy = \int -dy$$

$$g(y) = -y$$

$$f(x,y) = x^2y - y = c$$

Example: Solve

$$(e^{2y} - ycosxy)dx + (2xe^{2y} - xcos\,xy + 2y)dy = 0$$

$$M(x,y) = e^{2y} - ycosxy \; and \; N(x,y) = 2xe^{2y} - xcos\,xy + 2y$$

$$\frac{\partial M}{\partial y} = 2e^{2y} - cos\,xy - xy\,sin\,xy = \frac{\partial N}{\partial x} \quad exact$$

$$M(x,y) = e^{2y} - ycosxy = \; and \; N(x,y) = 2xe^{2y} - xcos\,xy + 2y$$

$$f(x,y) = \int (2xe^{2y} - xcos\,xy + 2y)dy + h(x) = xe^{2y} - sin\,xy + y^2 + h(x)$$

$$h'(x) = (e^{2y} - y\,cos\,xy) - (e^{2y} - ycosxy) \Longrightarrow \int h'(x)\,dx = \int 0\,dx$$

$$h(x) = c$$

$$f(x,y) = xe^{2y} - sin\,xy + y^2 + c$$

Example: Solve IVP

$$\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1 - x^2)}, \quad y(0) = 2$$

$$(\cos x \sin x - xy^2)dx + \; y(1 - x^2)dy = 0$$

$$M(x,y) = \cos x \sin x - xy^2 \; and \; N(x,y) = y(1 - x^2)$$

$$\frac{\partial M}{\partial y} = -2xy = \frac{\partial N}{\partial x} \quad exact$$

$$M(x, y) = \cos x \sin x - xy^2 = \; and \; N(x, y) = y\left(1 - x^2\right)$$

$$f(x, y) = \int (y(1 - x^2))dy + h(x) = \frac{y^2}{2}(1 - x^2) + h(x)$$

$$h'(x) = (\cos x \sin x - xy^2) + xy^2 \Longrightarrow \int h'(x)\, dx = \int \cos x \sin x \; dx$$

$$h(x) = -\frac{1}{2} cos\,^2 x$$

$$\frac{y^2}{2}(1 - x^2) - \frac{1}{2} cos\,^2 x = c_1$$

$$y^2(1 - x^2) - cos\,^2 x = c$$

Substitute the initial condition *c=3*. An implicit solution is

$$2^2(1 - 0^2) - cos\,^2(0) = c \Longrightarrow c = 3$$

$$y^2(1 - x^2) - cos\,^2 x = 3$$

$$y^2(1 - x^2) - cos\,^2 x = c$$