Average and r.m.s. values of sinusoidal currents and voltages

If I_m is the maximum value of a current which varies sinusoidally as shown in Fig. 3.9(a), the instantaneous value i is represented by: $i = I_m \sin\theta$. where θ is the angle in radians from instant of zero current.

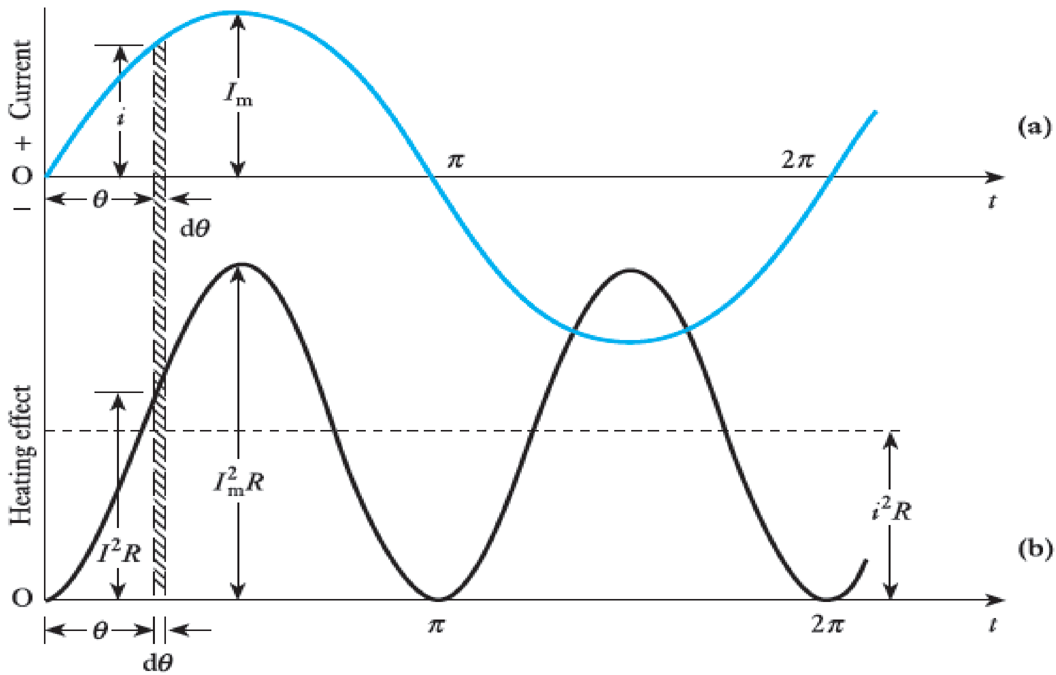


Fig. 3.9. Average and r.m.s. values of a sinusoidal current

The total area enclosed by the current wave over half-cycle is:

$$\int_0^\pi i \cdot d\theta = I_m \int_0^\pi \sin\theta \cdot d\theta = -I_m[\cos\theta]_0^\pi$$

$$= -I_m[-1-1] = 2I_m \text{ ampere radians}$$

Average value=Area under the curve/Base

From expression below, the average value of current over a half-cycle is:

$$\frac{2I_m\text{ [ampere radians]}}{\pi\text{ [radians]}}$$

i.e. $I_{av} = 0.637 I_m$ amperes

The average heating effect is**:[H.W]**

$$\frac{(\pi/2)I_m^2 R\text{ [watt radians]}}{\pi\text{ [radians]}} = \frac{1}{2} I_m^2 R \text{ watts}$$

r.m.s. value of a sinusoidal current or voltage is:

$$I = \frac{I_m}{\sqrt{2}} = 0.707 I_m$$

Form factor of a wave is

$$\frac{\text{RMS value}}{\text{Average value}}$$

Peak or crest factor of a wave is

$$\frac{\text{Peak or maximum value}}{\text{RMS value}}$$

Form factor of a sine wave is:

$$\frac{0.707 \times \text{maximum value}}{0.637 \times \text{maximum value}}$$

$$k_f = 1.11$$

and peak or crest factor of a sine wave is

$$\frac{\text{maximum value}}{0.707 \times \text{maximum value}}$$

$$\therefore \quad k_p = 1.414$$

Example2: An alternating voltage has the equation v = 141.4 sin 377t; what are the values of:

(a) r.m.s. voltage;

(b) frequency;

(c) the instantaneous voltage when t = 3 ms?

Sol: The relation is of the form $v = V_m \sin \omega t$ and, by comparison,

(a) $V_m = 141.4 \text{ V} = \sqrt{2} V$

hence $V = \frac{141.4}{\sqrt{2}} = \mathbf{100\ V}$

(b) Also by comparison

$\omega = 377 \text{ rad/s} = 2\pi f$

hence $f = \frac{377}{2\pi} = \mathbf{60\ Hz}$

(c) Finally

$v = 141.4 \sin 377t$

When $t = 3 \times 10^{-3}$ s

$v = 141.4 \sin(377 \times 3 \times 10^{-3}) = 141.4 \sin 1.131$

$= 141.4 \times 0.904 = \mathbf{127.8\ V}$

Average and r.m.s. values of non-sinusoidal currents and voltages

This can easily be done by considering this example.

Example3: A current has the following steady values in amperes for equal intervals of time changing instantaneously from one value to the next (Fig. 3.10):

0, 10, 20, 30, 20, 10, 0, −10, −20, −30, −20, −10, 0, etc.

Calculate the r.m.s. value of the current and its form factor.

Sol:

Because of the symmetry of the waveform, it is only necessary to calculate the values over the first half-cycle.

If n equidistant mid-ordinates, i_1, i_2, etc. are taken over either the positive or the negative half-cycle, then *average* value of current over half a cycle is:

$$I_{av} = \frac{i_1 + i_2 + \ldots + i_n}{n}$$

The root-mean-square (or r.m.s.) value of the current is:

$$I = \sqrt{\left(\frac{i_1^2 + i_2^2 + \ldots + i_n^2}{n}\right)}$$

$$I_{av} = \frac{\text{area under curve}}{\text{length of base}}$$

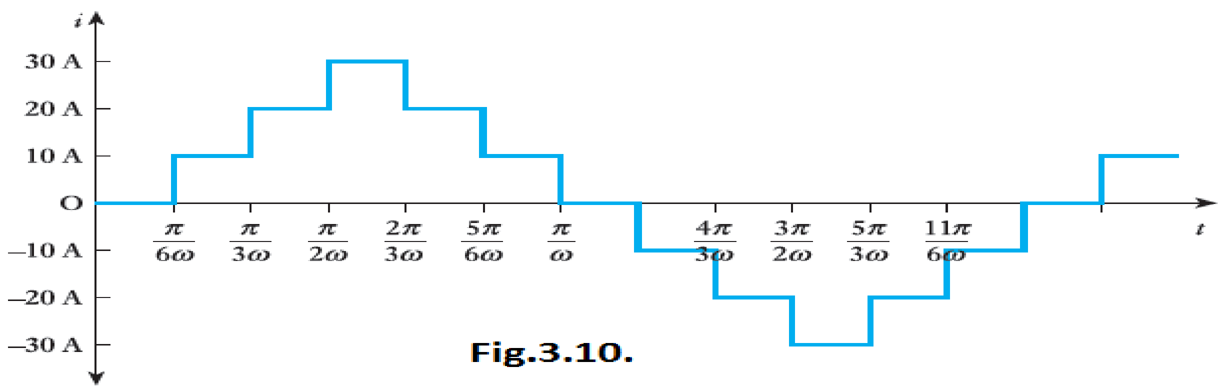


Fig.3.10.

$$= \frac{0\left(\frac{\pi}{6\omega} - 0\right) + 10\left(\frac{2\pi}{6\omega} - \frac{\pi}{6\omega}\right) + 20\left(\frac{3\pi}{6\omega} - \frac{2\pi}{6\omega}\right) + 30\left(\frac{4\pi}{6\omega} - \frac{3\pi}{6\omega}\right) + 20\left(\frac{5\pi}{6\omega} - \frac{4\pi}{6\omega}\right) + 10\left(\frac{6\pi}{6\omega} - \frac{5\pi}{6\omega}\right)}{\frac{\pi}{\omega} - 0}$$

$$= 15.0\text{ A}$$

$$I^2 = \frac{0^2\left(\frac{\pi}{6\omega} - 0\right) + 10^2\left(\frac{2\pi}{6\omega} - \frac{\pi}{6\omega}\right) + 20^2\left(\frac{3\pi}{6\omega} - \frac{2\pi}{6\omega}\right) + 30^2\left(\frac{4\pi}{6\omega} - \frac{3\pi}{6\omega}\right) + 20^2\left(\frac{5\pi}{6\omega} - \frac{4\pi}{6\omega}\right) + 10^2\left(\frac{6\pi}{6\omega} - \frac{5\pi}{6\omega}\right)}{\frac{\pi}{\omega} - 0}$$

$$= 316$$

$$I = \sqrt{316} = 17.8\text{ A}$$

$$k_f = \frac{I}{I_{av}} = \frac{17.8}{15.0} = 1.19$$

A.C. Through Resistance, Inductance and Capacitance

We will now consider the phase angle introduced between an alternating voltage and current when the circuit contains resistance only, inductance only and capacitance only. ***In each case, we will assume that we are given the alternating voltage of equation $e = E_m \sin \omega t$*** and will proceed to find the equation and the phase of the alternating current produced in each case.

A.C. Through Pure Ohmic Resistance Alone

The circuit is shown in Fig. 3.11. Let the applied voltage be given by the equation.

$$v = V_m \sin \theta = V_m \sin \omega t \qquad ...(i)$$

Let R = ohmic resistance ; i = instantaneous current

Obviously, the applied voltage has to supply ohmic voltage drop only. Hence

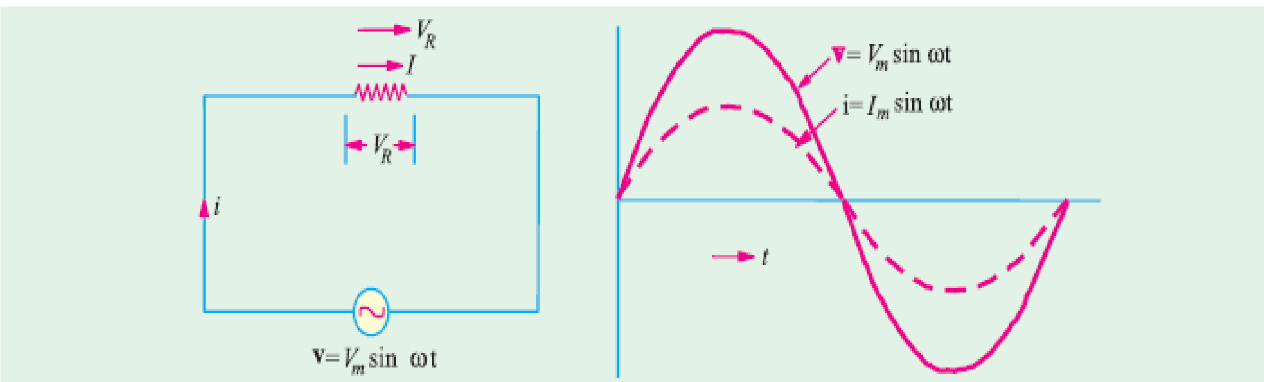
$$v = iR;$$

Putting the value of 'v' from above, we get $V_m \sin \omega t = iR$; $i = \frac{V_m}{R} \sin \omega t$ ...(ii)

Current 'i' is maximum when $\sin \omega t$ is unity $\therefore I_m = V_m/R$ Hence, equation (ii) becomes,

$$i = I_m \sin \omega t \qquad ...(iii)$$

Comparing (i) and (ii), we find that the alternating voltage and current are in phase with each other as shown in Fig. 3.12.



Power. Instantaneous power, $p = vi = V_m I_m \sin^2 \omega t$ ...(Fig. 3.13)

$$= \frac{V_m I_m}{2}(1 - \cos 2\omega t) = \frac{V_m I_m}{2} - \frac{V_m I_m}{2} \cos 2\omega t$$

Power consists of a constant part $\frac{V_m I_m}{2}$ and a fluctuating part $\frac{V_m I_m}{2} \cos 2\omega t$ of frequency double that of voltage and current waves. For a complete cycle. the average value of $\frac{V_m I_m}{2} \cos 2\omega t$ is zero.

Hence, power for the whole cycle is

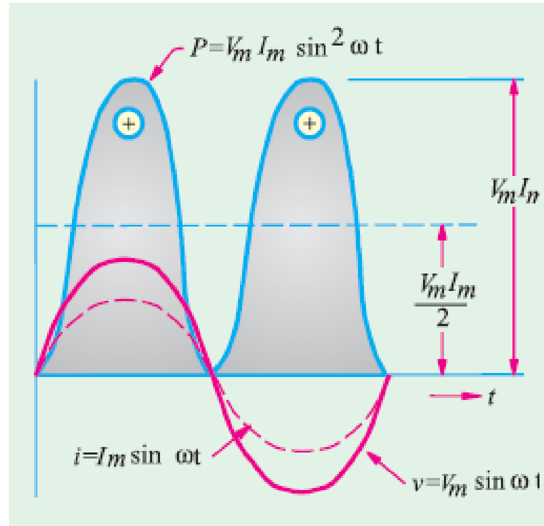
$$P = \frac{V_m I_m}{2} = \frac{V_m}{\sqrt{2}} \times \frac{I_m}{\sqrt{2}}$$

or $\quad P = V \times I$ watt

where $\quad V$ = r.m.s. value of applied voltage.

$\qquad I$ = r.m.s. value of the current.

It is seen from (Fig. 3.13) that no part of the power cycle becomes negative at any time. In other words, in a purely resistive circuit, power is never zero. This is so because the instantaneous values of voltage and current are always either both positive or negative and hence the product is always positive.



(Fig. 3.13)

A.C. Through Pure Inductance Alone

Whenever an alternating voltage is applied to a purely inductive coil, a back e.m.f. is produced due to the self-inductance of the coil. The back e.m.f., at every step, opposes the rise or fall of current through the coil. As there is no ohmic voltage drop, the applied voltage has to overcome this self-induced e.m.f. only. So at every step

$$v = L\frac{di}{dt}$$

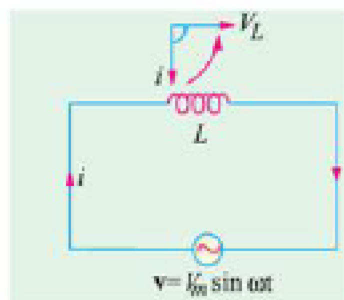
Now $\quad v = V_m \sin \omega t$

$\therefore \quad V_m \sin \omega t = L\frac{di}{dt} \quad \therefore \quad di = \frac{V_m}{L}\sin \omega t\, dt$

Integrating both sides, we get $i = \frac{V_m}{L}\int \sin \omega t\, dt$

$$\frac{V_m}{\omega L}(\quad \cos \omega t)$$

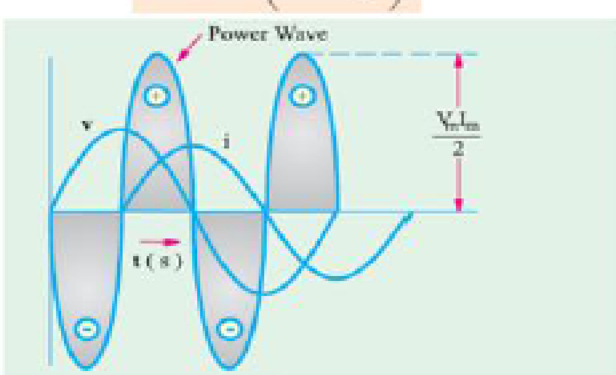
$\therefore \quad = \frac{V_m}{\omega L}\sin\left(\omega t - \frac{\pi}{2}\right) = \frac{V_m}{X_L}\sin(\omega t - \pi/2)$



Max. value of i is $I_m = \frac{V_m}{\omega L}$ when $\sin\left(\omega t - \frac{\pi}{2}\right)$ is unity.

Hence, the equation of the current becomes $i = I_m \sin(\omega t - \pi/2)$.

So, we find that if applied voltage is represented by $v = V_m \sin \omega t$, then current flowing in a *purely* inductive circuit is given by $i = I_m \sin\left(\omega t - \frac{\pi}{2}\right)$



Clearly, the current lags behind the applied voltage by the phase difference between the two is π/2 with voltage leading. Vectors are shown in Fig. 3.14 where voltage has been taken along the reference axis. We have seen that $I_m = V_m/\omega L = V_m/X_L$. Here '$\omega L$' plays the part of 'resistance'. It is called the (inductive) ***reactance*** X_L of the coil and is given in ohms if L is in henry and ω is in radian/second.

Now, $X_L = \omega L = 2\pi f L$ ohm. It is seen that X_L depends directly on frequency of the voltage. Higher the value of f, greater the reactance offered and ***vice-versa***.

Instantaneous power = vi = $V_m I_m \sin \omega t . \cos \omega t = 0.5 V_m I_m \sin(2\omega t)$.

Power for whole cycle is $P = \frac{V_m I_m}{2} \int_0^{2\pi} \sin 2\omega t \, dt = 0$

A.C. Through Pure Capacitance Alone

When an alternating voltage is applied to the plates of a capacitor, the capacitor is charged first in one direction and then in the opposite direction. When reference to Fig. 3.16. , let

v = p.d. developed between plates at any instant

q = Charge on plates at that instant.

Then $q = Cv$ ...where C is the capacitance

$= C V_m \sin \omega t$ ...putting the value of v.

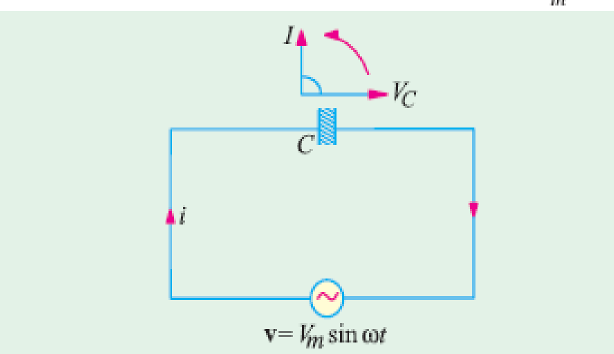


Fig.3.16

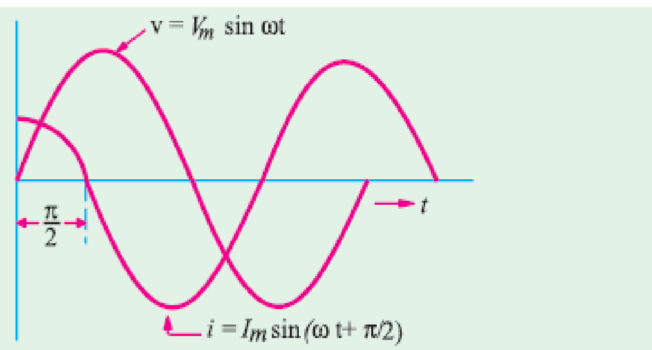


Fig.3.17

Now, current i is given by the rate of flow of charge.

$$\therefore \quad i = \frac{dq}{dt} = \frac{d}{dt}(CV_m \sin \omega t) \quad CV_m \cos \omega t \text{ or } i = \frac{V_m}{1/\omega C} \cos \omega t \quad \frac{V_m}{1/\omega C} \sin \omega \, t + \frac{\pi}{2}$$

Obviously, $I_m = \frac{V_m}{1/\omega C} = \frac{V_m}{X_C}$ $i = I_m \sin \omega t + \frac{\pi}{2}$

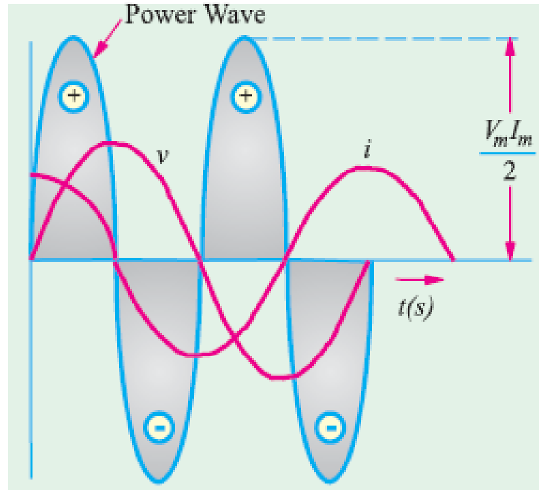


Fig.3.18.

The denominator $X_C = 1/\omega C$ is known as capacitive *reactance* and is in ohms if C is in farad and ω in radian/second. It is seen that if the applied voltage is given by $v = V_m \sin \omega t$, then the current is given by $i = I_m \sin (\omega t + \pi/2)$.

Hence, we find that the current in a pure capacitor leads its voltage by a quarter cycle as shown in Fig. 3.17. or phase difference between its voltage and current is $\pi/2$ with the current leading. Vector representation is given in Fig. 3.17.. Note that V_c is taken along the reference axis.

Power. Instantaneous power

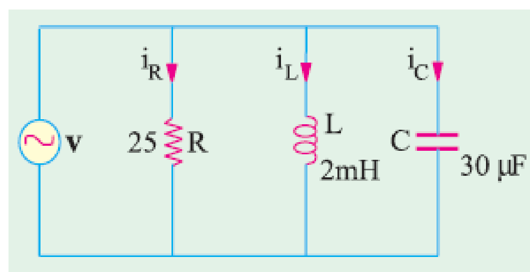
$$p = vi = V_m \sin \omega t . I_m \sin (\omega t + 90°)$$

$$= V_m I_m \sin \quad t \cos \quad t \quad \frac{1}{2} V_m I_m \sin 2 \quad t$$

Power for the whole cycle

$$= \frac{1}{2} V_m I_m \int_0^{2\pi} \sin 2\omega t \, dt = 0$$

Example *The voltage applied across 3-branched circuit of Fig.* below *is given by $v = 100 \sin (5000t + \pi/4)$. Calculate the branch currents and total current.*



Solution. The total instantaneous current is the vector sum of the three branch currents.

$$i_t = i_R + i_L + i_C.$$

Now $i_R = v/R = 100 \sin (5000\,t + \pi/4)/25$

$$= 4 \sin (5000\,t + \pi/4)$$

$$i_L = \frac{1}{L} \int v \, dt = \frac{10^3}{2} \int 100 \sin \left(5000t + \frac{\pi}{4}\right) dt$$

$$= \frac{10^3 \times 100}{2} \left[\frac{-\cos (5000\,t + \pi/4)}{5000} \right] = -10 \cos (5000\,t + \pi/4)$$

$$i_C = C \frac{dv}{dt} = C . \frac{d}{dt} [100 \sin (5000\,t + \pi/4)]$$

$$= 30 \times 10^{-6} \times 100 \times 5000 \times \cos (5000\,t + \pi/4) = 15 \cos (5000\,t + \pi/4)$$

$$i_t = 4 \sin (5000\,t + \pi/4) - 10 \cos (5000\,t + \pi/4) + 15 \cos (5000t + \pi/4)$$

$$= \mathbf{4 \sin (5000\,t + \pi/4) + 5 \cos (5000\,t + \pi/4)}$$