Chapter Three

A.C. Circuits

Introduction

In previous chapter we have considered circuits and networks in which the current has remained constant, i.e. direct current systems. However, there remains another type of system – the alternating system – in which the magnitudes of the voltage and of the current vary in a repetitive manner. Examples of such repetitive currents are shown in Fig. 3.1.

An alternating quantity is one that regularly acts first in one direction and then in the opposite direction and do not have constant magnitude with time. Its magnitude continuously vary with time.

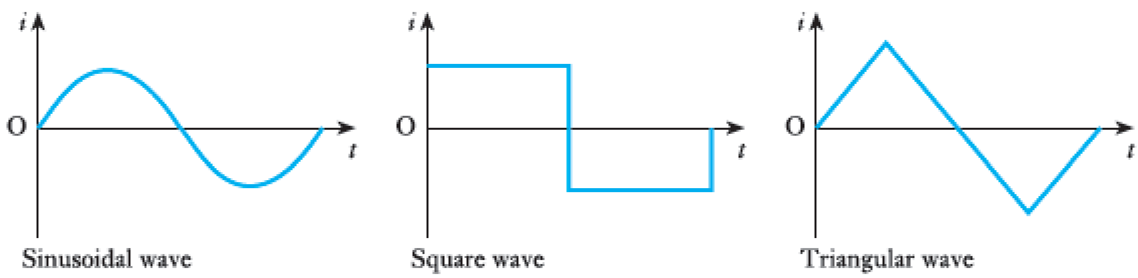


Fig. 3.1 Alternating waveforms

Alternating current can be abbreviated to a.c., hence a system with such an alternating current is known as an a.c. system. The above curves relating current to time are known as waveforms.

Generation of an alternating e.m.f.

Fig.3.2 shows a loop AB carried by a spindle DD rotated at a constant speed in an anticlockwise direction in a uniform magnetic field due to poles NS. The ends of the loop are brought out to two slip-rings C_1 and C_2, attached to DD. Bearing on these rings are carbon brushes E_1 and E_2, which are connected to an external resistor R. When the plane of the loop is horizontal, as shown in Fig. 3.3(a), the two sides A and B are moving parallel to the direction of the magnetic flux; it follows that no flux is being cut and no e.m.f. is being generated in the loop. Subsequent diagrams in Fig.3.3 show the effects which occur as the coil is rotated. In Fig. 3.3(b), the coil sides are cutting the flux and therefore an e.m.f. is induced in the coil sides. Since the coil sides are moving in opposite directions, the e.m.fs act in opposite directions, as shown by the dot and cross notation. However, in this case the e.m.f. which appears at the brushes is twice that which is induced in a coil side. Once the coil reaches the position shown in Fig. 3.3(c), the rate of cutting reaches a maximum. Thereafter the e.m.f. falls to zero by the time the coil has rotated to the position shown in Fig. 3.3(d).

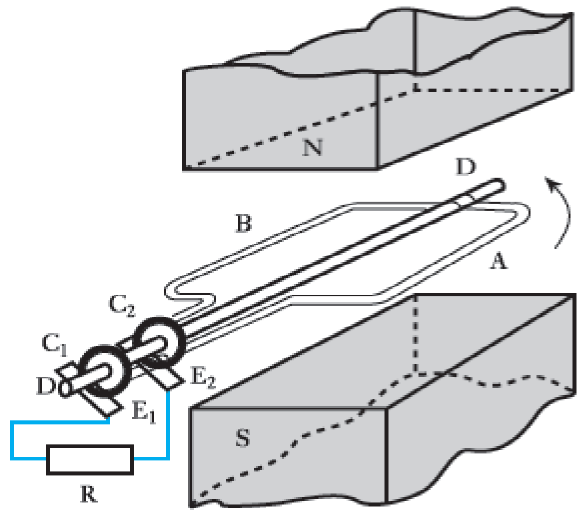


Fig. 3.2 Generation of an alternating e.m.f.

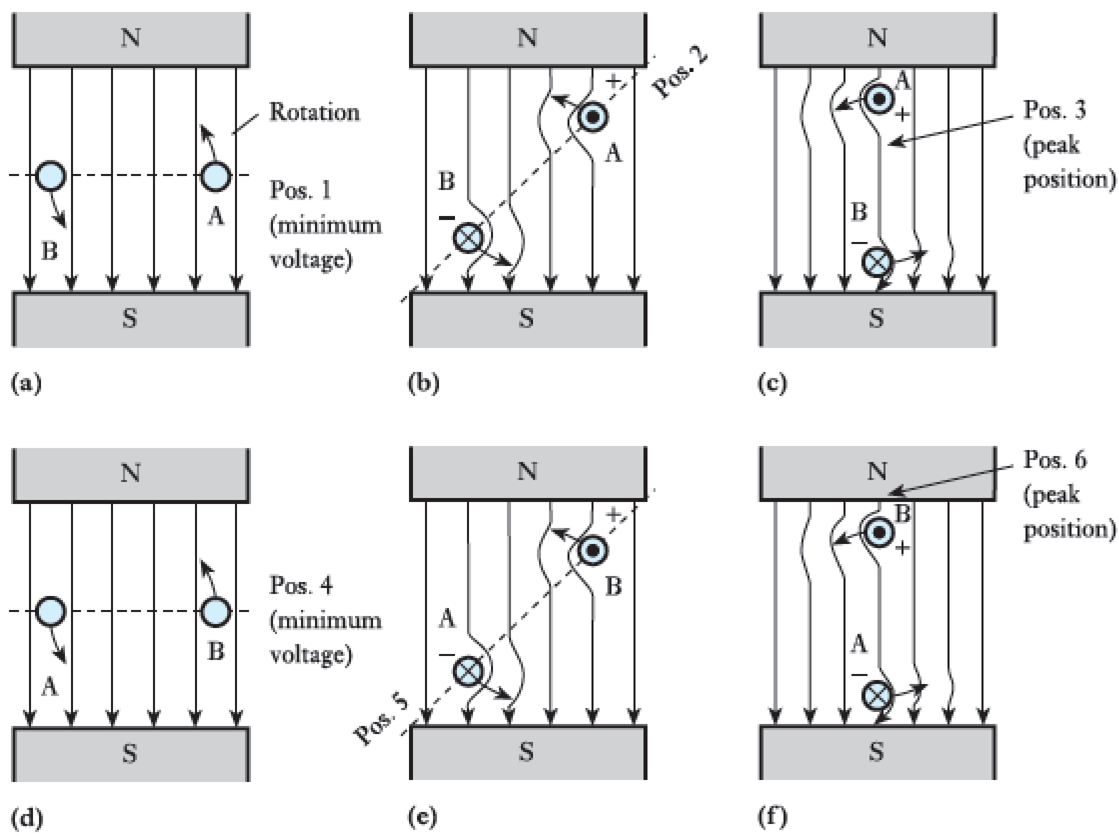
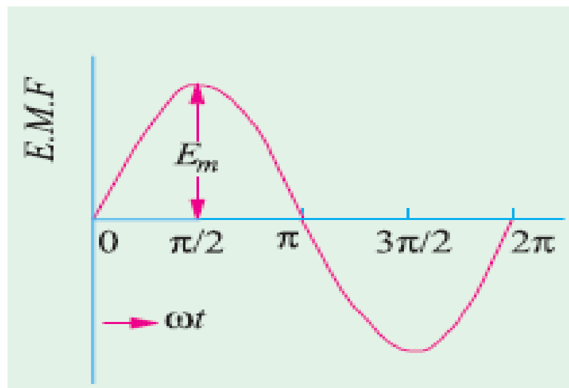


Fig. 3.3 EMF in rotating coil

The induced e.m.f. in the position shown in Fig. 3.3(e) is appear that the diagram is the same as that of Fig. 3.3(b), but in fact it is side A which bears the cross while side B has the dot. This means that the e.m.f. is of the same magnitude but of the opposite polarity. This observation also applies to Fig. 3.3(f). It follows that the variation of induced e.m.f. during the second half of the cycle of rotation is the same in magnitude as during the first half but the polarity of the e.m.f. has reversed.

It is seen that the induced e.m.f. varies as sine function of the time angle ωt and when e.m.f. is plotted against time, a curve similar to the one shown in Fig. 3.4 is obtained. This curve is known as sine curve and the e.m.f. which varies in this manner is known as sinusoidal e.m.f.

The e.m.f. generated in one side of the coil which contains N conductors, is given by, $e = N\,Bl\,v\sin\theta$ (volt).



Where:

N=Number of coil turns. B=Flux density (Wb./m^2). l=length of coil sides (meters). v=velocity (metre/second), θ= ωt.

ω=2πf.

f=frequency of rotation of the coil in Hz.

The total e.m.f. generated in loop is

$e = 2N\,Bl\,v\sin\theta$ (volt)……*(i)*

Now, e has maximum value of E_m (say) when θ = 90°. Hence, from Eq. (*i*) above, we get, $E_m = 2\,B\,N\,l\,v$ volt. Therefore Eq. (*i*) can be rewritten as $e = E_m \sin\theta$

If b = width of the coil in meters ; f = frequency of rotation of coil in Hz, then $v = \pi\,bf$

$$\therefore \quad E_m = 2\,B\,N\,l \times \pi\,b\,f = 2\,\pi\,f\,N\,B\,A \text{ volts}$$

Similarly, the equation of induced alternating current is $i = I_m \sin\omega t$

Since ω= 2πf, where f is the frequency of rotation of the coil, the above equations of the voltage and current can be written as

$$e = E_m \sin 2\,\pi\,f\,t = E_m \sin\left(\frac{2\pi}{T}\right)t \quad \text{and} \quad i = I_m \sin 2\,\pi\,ft = I_m \sin\left(\frac{2\pi}{T}\right)t$$

where T = time-period of the alternating voltage or current = $1/f$

Waveform terms and definitions

Waveform. The variation of a quantity such as voltage or current shown on a graph to a base of time.

Cycle. One complete set of positive and negative values of alternating quantity is known as cycle.

Period. The time taken by an alternating quantity to complete one cycle is called its time period T. Fig.3.6. Illustrates a variety of situations in which the cycle and period have identical values.

Instantaneous value. The magnitude of a waveform at any instant in time. Instantaneous values are denoted by lower-case symbols such as e, v and i.

Peak value. The maximum instantaneous value measured from its zero value is known as its peak value.

Peak-to-peak value. The maximum variation between the maximum positive instantaneous value and the maximum negative instantaneous value is the peak-to-peak value. For a sinusoidal waveform, this is twice the peak value. The peak-to-peak value is E_{pp} or V_{pp} or I_{pp}.

The relationships between peak value and peak-to-peak value are illustrated in Fig. 3.7.

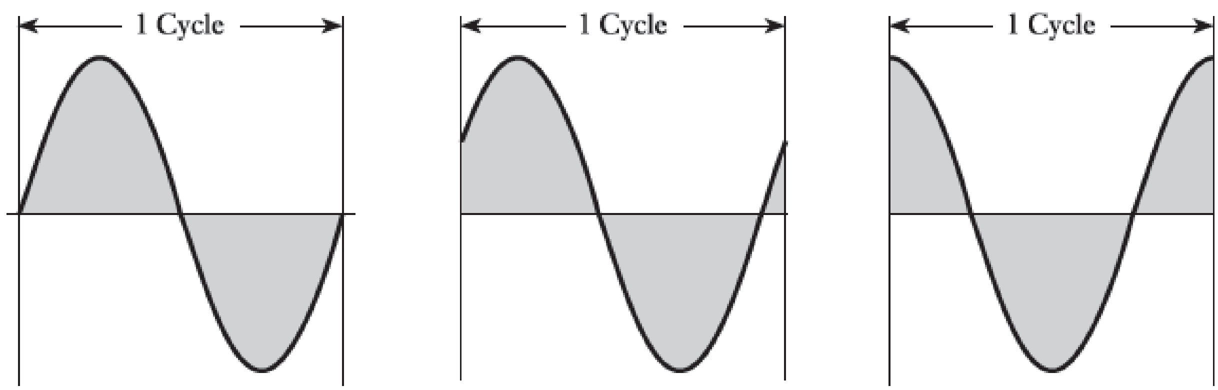


Fig. 3.6 Cycles and periods

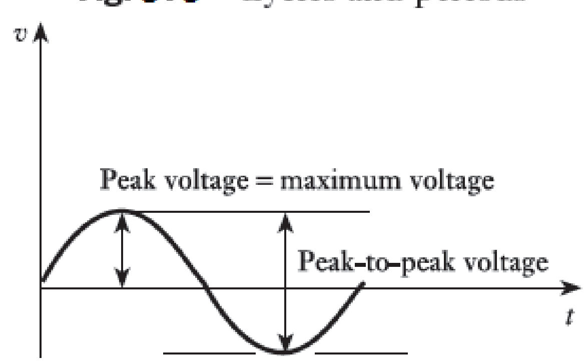


Fig. 3.7 Peak values

Frequency. The number of cycles that occur in 1 second is termed the frequency of that quantity. Frequency is measured in hertz (Hz). It follows that frequency f is related to the period T by the relation:

$$f = \frac{1}{T}$$

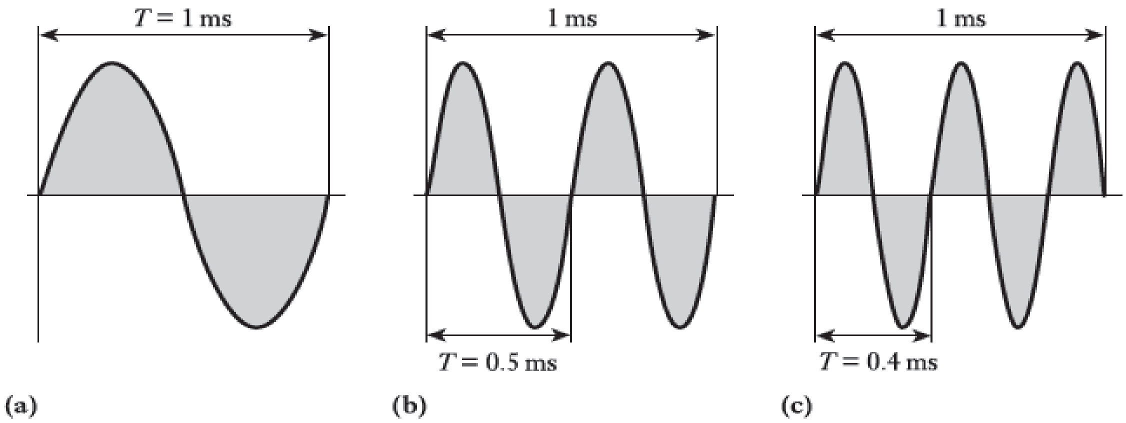


Fig. 3.8. Effect on waveforms by varying frequency

Example1: A coil of 100 turns is rotated at 1500 r/min in a magnetic field having a uniform density of 0.05 T, the axis of rotation being at right angles to the direction of the flux. The area per turn is 40 cm^2. Calculate

(a) the frequency;

(b) the period;

(c) the maximum value of the generated e.m.f.;

(d) the value of the generated e.m.f. when the coil has rotated through 30° from the position of zero e.m.f.

Sol:

(a) Since the e.m.f. generated in the coil undergoes one cycle of variation when the coil rotates through one revolution,

$$\therefore \quad \text{Frequency} = \text{no. of cycles per second}$$
$$= \text{no. of revolutions per second}$$
$$= \frac{1500}{60} = 25 \text{ Hz}$$

(b)
$$\text{Period} = \text{time of 1 cycle}$$
$$= \frac{1}{25} = 0.04 \text{ s}$$

(c) $E_m = 2\pi \times 0.05 \times 0.004 \times 100 \times 1500/60 = 3.14$ V

(d) For $\theta = 30°$, $\sin 30° = 0.5$, $\therefore\ e = 3.14 \times 0.5 = 1.57$ V

Average and r.m.s. values of sinusoidal currents and voltages

If I_m is the maximum value of a current which varies sinusoidally as shown in Fig. 3.9(a), the instantaneous value i is represented by: $i = I_m \sin\theta$. where θ is the angle in radians from instant of zero current.

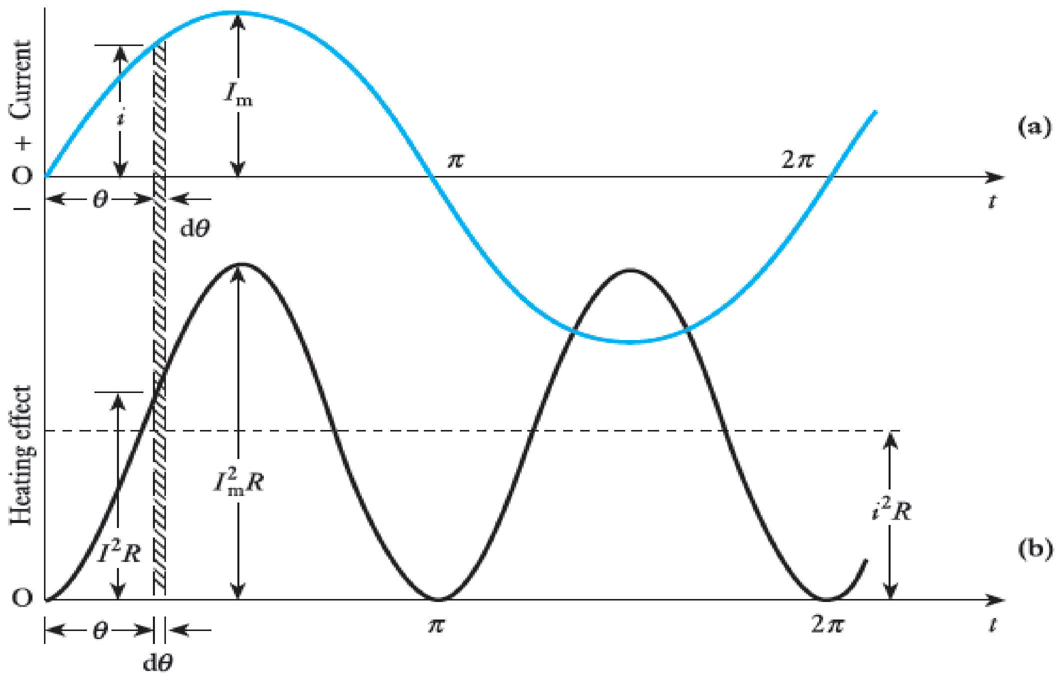


Fig. 3.9. Average and r.m.s. values of a sinusoidal current

The total area enclosed by the current wave over half-cycle is:

$$\int_0^{\pi} i \cdot d\theta = I_m \int_0^{\pi} \sin\theta \cdot d\theta = -I_m \left[\cos\theta\right]_0^{\pi}$$

$$= -I_m[-1 - 1] = 2I_m \text{ ampere radians}$$

Average value=Area under the curve/Base

From expression below, the average value of current over a half-cycle is:

$$\frac{2I_m \text{ [ampere radians]}}{\pi \text{ [radians]}}$$

i.e. $I_{av} = 0.637 I_m$ amperes

The average heating effect is**:[H.W]**

$$\frac{(\pi/2) I_m^2 R \text{ [watt radians]}}{\pi \text{ [radians]}} = \tfrac{1}{2} I_m^2 R \text{ watts}$$

r.m.s. value of a sinusoidal current or voltage is:

$$I = \frac{I_m}{\sqrt{2}} = 0.707 I_m$$

Form factor of a sine wave is:

$$\frac{0.707 \times \text{maximum value}}{0.637 \times \text{maximum value}}$$

$$k_f = 1.11$$

and peak or crest factor of a sine wave is

$$\frac{\text{maximum value}}{0.707 \times \text{maximum value}}$$

$$\therefore \quad k_p = 1.414$$

Form factor of a wave is

$$\frac{\text{RMS value}}{\text{Average value}}$$

Peak or *crest factor* of a wave is

$$\frac{\text{Peak or maximum value}}{\text{RMS value}}$$

Example2: An alternating voltage has the equation v = 141.4 sin 377t; what are the values of:

(a) r.m.s. voltage;

(b) frequency;

(c) the instantaneous voltage when t = 3 ms?

Sol: The relation is of the form $v = V_m \sin \omega t$ and, by comparison,

(a) $V_m = 141.4 \text{ V} = \sqrt{2} V$

hence $V = \frac{141.4}{\sqrt{2}} = \mathbf{100 \ V}$

(b) Also by comparison

$\omega = 377 \text{ rad/s} = 2\pi f$

hence $f = \frac{377}{2\pi} = \mathbf{60 \ Hz}$

(c) Finally

$v = 141.4 \sin 377t$

When $t = 3 \times 10^{-3}$ s

$v = 141.4 \sin(377 \times 3 \times 10^{-3}) = 141.4 \sin 1.131$

$= 141.4 \times 0.904 = \mathbf{127.8 \ V}$

Average and r.m.s. values of non-sinusoidal currents and voltages

This can easily be done by considering this example.

Example3: A current has the following steady values in amperes for equal intervals of time changing instantaneously from one value to the next (Fig. 3.10):

0, 10, 20, 30, 20, 10, 0, −10, −20, −30, −20, −10, 0, etc.

Calculate the r.m.s. value of the current and its form factor.

Sol:

Because of the symmetry of the waveform, it is only necessary to calculate the values over the first half-cycle.

If n equidistant mid-ordinates, i_1, i_2, etc. are taken over either the positive or the negative half-cycle, then *average* value of current over half a cycle is:

$$I_{av} = \frac{i_1 + i_2 + \ldots + i_n}{n}$$

The root-mean-square (or r.m.s.) value of the current is:

$$I = \sqrt{\left(\frac{i_1^2 + i_2^2 + \ldots + i_n^2}{n}\right)}$$

$$I_{av} = \frac{\text{area under curve}}{\text{length of base}}$$

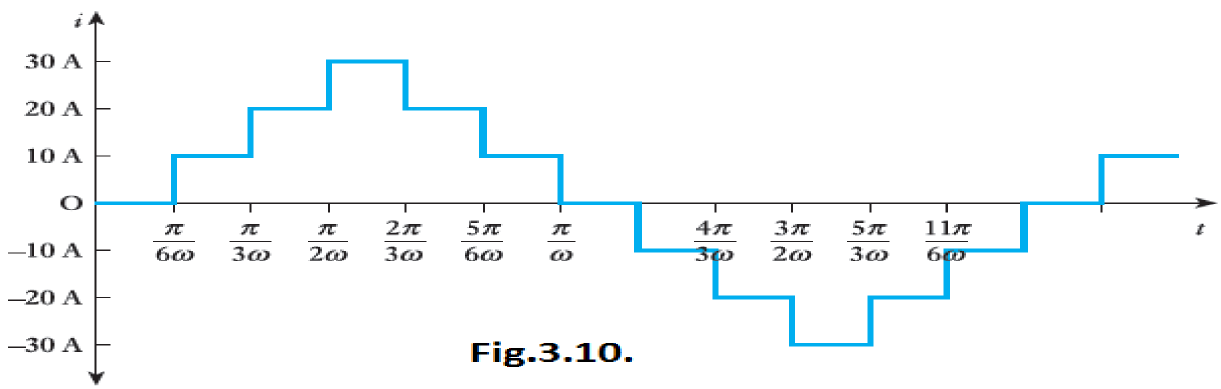


Fig.3.10.

$$= \frac{0\left(\frac{\pi}{6\omega} - 0\right) + 10\left(\frac{2\pi}{6\omega} - \frac{\pi}{6\omega}\right) + 20\left(\frac{3\pi}{6\omega} - \frac{2\pi}{6\omega}\right) + 30\left(\frac{4\pi}{6\omega} - \frac{3\pi}{6\omega}\right) + 20\left(\frac{5\pi}{6\omega} - \frac{4\pi}{6\omega}\right) + 10\left(\frac{6\pi}{6\omega} - \frac{5\pi}{6\omega}\right)}{\frac{\pi}{\omega} - 0}$$

$$= 15.0 \text{ A}$$

$$I^2 = \frac{0^2\left(\frac{\pi}{6\omega} - 0\right) + 10^2\left(\frac{2\pi}{6\omega} - \frac{\pi}{6\omega}\right) + 20^2\left(\frac{3\pi}{6\omega} - \frac{2\pi}{6\omega}\right) + 30^2\left(\frac{4\pi}{6\omega} - \frac{3\pi}{6\omega}\right) + 20^2\left(\frac{5\pi}{6\omega} - \frac{4\pi}{6\omega}\right) + 10^2\left(\frac{6\pi}{6\omega} - \frac{5\pi}{6\omega}\right)}{\frac{\pi}{\omega} - 0}$$

$$= 316$$

$$I = \sqrt{316} = 17.8 \text{ A}$$

$$k_f = \frac{I}{I_{av}} = \frac{17.8}{15.0} = 1.19$$

A.C. Through Resistance, Inductance and Capacitance

We will now consider the phase angle introduced between an alternating voltage and current when the circuit contains resistance only, inductance only and capacitance only. ***In each case, we will assume that we are given the alternating voltage of equation $e = E_m \sin \omega t$*** and will proceed to find the equation and the phase of the alternating current produced in each case.

A.C. Through Pure Ohmic Resistance Alone

The circuit is shown in Fig. 3.11. Let the applied voltage be given by the equation.

$$v = V_m \sin \theta = V_m \sin \omega t \quad \text{...}(i)$$

Let R = ohmic resistance ; i = instantaneous current

Obviously, the applied voltage has to supply ohmic voltage drop only. Hence

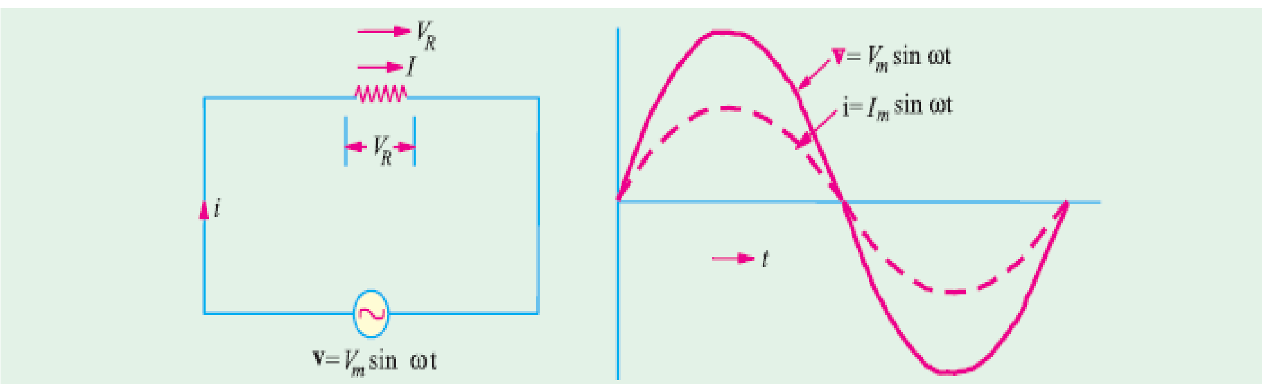
$$v = iR;$$

Putting the value of 'v' from above, we get $V_m \sin \omega t = iR$; $i = \frac{V_m}{R} \sin \omega t$ ...(ii)

Current 'i' is maximum when $\sin \omega t$ is unity $\therefore I_m = V_m/R$ Hence, equation (ii) becomes,

$$i = I_m \sin \omega t \quad \text{...}(iii)$$

Comparing (i) and (ii), we find that the alternating voltage and current are in phase with each other as shown in Fig. **3.12.**



Power. Instantaneous power, $p = vi = V_m I_m \sin^2 \omega t$ ...(Fig. **3.13**)

$$= \frac{V_m I_m}{2}(1 - \cos 2\omega t) = \frac{V_m I_m}{2} - \frac{V_m I_m}{2} \cos 2\omega t$$

Power consists of a constant part $\frac{V_m I_m}{2}$ and a fluctuating part $\frac{V_m I_m}{2} \cos 2\omega t$ of frequency double that of voltage and current waves. For a complete cycle. the average value of $\frac{V_m I_m}{2} \cos 2\omega t$ is zero.

Hence, power for the whole cycle is

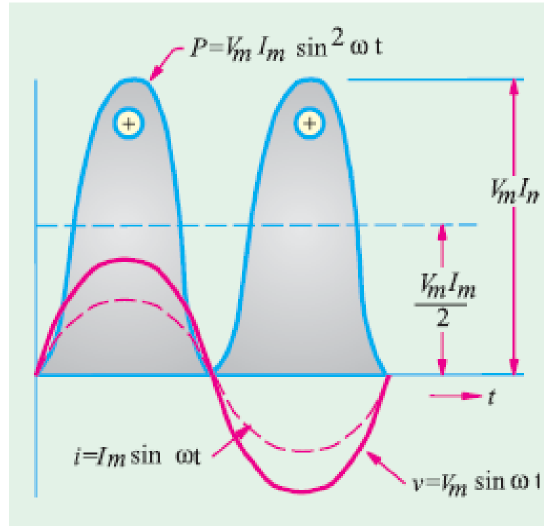
$$P = \frac{V_m I_m}{2} = \frac{V_m}{\sqrt{2}} \times \frac{I_m}{\sqrt{2}}$$

or $\quad P = V \times I$ watt

where $\quad V$ = r.m.s. value of applied voltage.

$\quad I$ = r.m.s. value of the current.

It is seen from (Fig. 3.13) that no part of the power cycle becomes negative at any time. In other words, in a purely resistive circuit, power is never zero. This is so because the instantaneous values of voltage and current are always either both positive or negative and hence the product is always positive.



(Fig. 3.13)

A.C. Through Pure Inductance Alone

Whenever an alternating voltage is applied to a purely inductive coil, a back e.m.f. is produced due to the self-inductance of the coil. The back e.m.f., at every step, opposes the rise or fall of current through the coil. As there is no ohmic voltage drop, the applied voltage has to overcome this self-induced e.m.f. only. So at every step

$$v = L\frac{di}{dt}$$

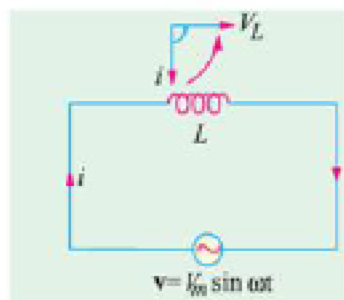
Now $\quad v = V_m \sin \omega t$

$\therefore \quad V_m \sin \omega t = L\frac{di}{dt} \quad \therefore \quad di = \frac{V_m}{L}\sin \omega t \, dt$

Integrating both sides, we get $i = \frac{V_m}{L}\int \sin \omega t \, dt$

$$\frac{V_m}{\omega L}(\ \cos \omega t)$$

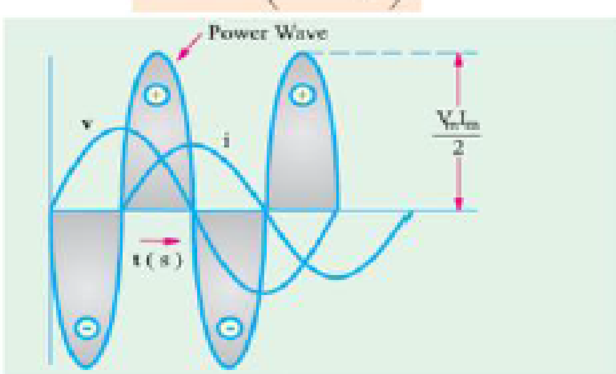
$\therefore \quad = \frac{V_m}{\omega L}\sin\left(\omega t - \frac{\pi}{2}\right) = \frac{V_m}{X_L}\sin(\omega t - \pi/2)$



Max. value of i is $I_m = \frac{V_m}{\omega L}$ when $\sin\left(\omega t - \frac{\pi}{2}\right)$ is unity.

Hence, the equation of the current becomes $i = I_m \sin(\omega t - \pi/2)$.

So, we find that if applied voltage is represented by $v = V_m \sin \omega t$, then current flowing in a *purely* inductive circuit is given by $i = I_m \sin\left(\omega t - \frac{\pi}{2}\right)$



Clearly, the current lags behind the applied voltage by the phase difference between the two is π/2 with voltage leading. Vectors are shown in Fig. 3.14 where voltage has been taken along the reference axis. We have seen that $I_m = V_m/\omega L = V_m/X_L$. Here 'ω*L*' plays the part of 'resistance'. It is called the (inductive) ***reactance*** X_L of the coil and is given in ohms if *L* is in henry and ω is in radian/second.

Now, $X_L = \omega L = 2\pi f L$ ohm. It is seen that X_L depends directly on frequency of the voltage. Higher the value of *f*, greater the reactance offered and ***vice-versa***.

Instantaneous power = v*i* = $V_m I_m \sin \omega t . \cos \omega t = 0.5 V_m I_m \sin(2\omega t)$.

Power for whole cycle is $P = \dfrac{V_m I_m}{2} \int_0^{2\pi} \sin 2\omega t \, dt = 0$

A.C. Through Pure Capacitance Alone

When an alternating voltage is applied to the plates of a capacitor, the capacitor is charged first in one direction and then in the opposite direction. When reference to Fig. 3.16. , let

v = p.d. developed between plates at any instant

q = Charge on plates at that instant.

Then $q = Cv$ ...where *C* is the capacitance

$= C V_m \sin \omega t$ ...putting the value of *v*.

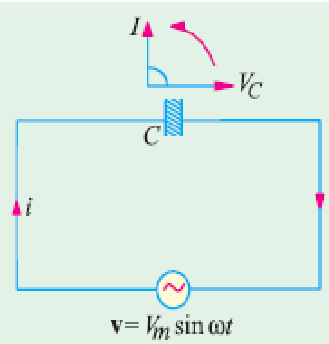


Fig.3.16

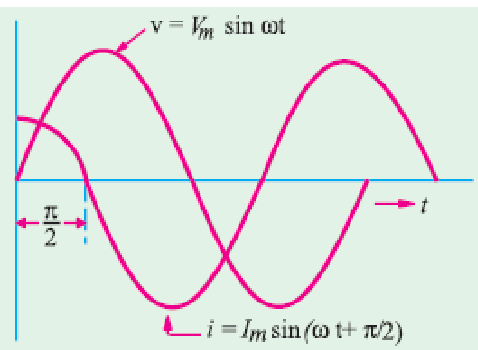


Fig.3.17

Now, current *i* is given by the rate of flow of charge.

$$\therefore \quad i = \frac{dq}{dt} = \frac{d}{dt}(CV_m \sin \omega t) \qquad CV_m \cos \omega t \text{ or } i = \frac{V_m}{1/\omega C} \cos \omega t \qquad \frac{V_m}{1/\omega C} \sin \omega \, t + \frac{\pi}{2}$$

Obviously, $I_m = \frac{V_m}{1/\omega C} = \frac{V_m}{X_C}$ $i = I_m \sin\ \omega t + \frac{\pi}{2}$

The denominator $X_C = 1/\omega C$ is known as capacitive *reactance* and is in ohms if C is in farad and ω in radian/second. It is seen that if the applied voltage is given by $v = V_m \sin \omega t$, then the current is given by $i = I_m \sin(\omega t + \pi/2)$.

Hence, we find that the current in a pure capacitor leads its voltage by a quarter cycle as shown in Fig. 3.17. or phase difference between its voltage and current is $\pi/2$ with the current leading. Vector representation is given in Fig. 3.17.. Note that V_c is taken along the reference axis.

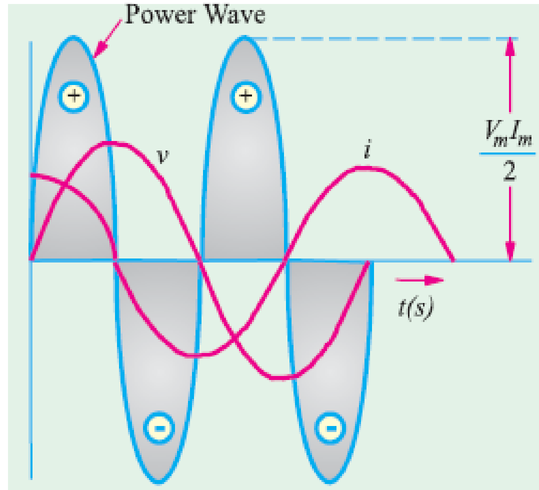


Fig.3.18.

Power. Instantaneous power

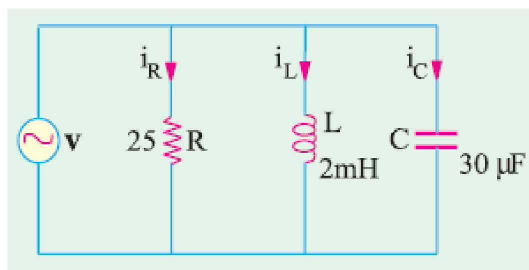
$$p = vi = V_m \sin \omega t . I_m \sin(\omega t + 90°)$$

$$= V_m I_m \sin\ \ t \cos\ \ t \quad \frac{1}{2} V_m I_m \sin 2\ \ t$$

Power for the whole cycle

$$= \frac{1}{2} V_m I_m \int_0^{2\pi} \sin 2\omega t\ dt = 0$$

Example *The voltage applied across 3-branched circuit of Fig.* below *is given by $v = 100 \sin(5000t + \pi/4)$. Calculate the branch currents and total current.*



Solution. The total instantaneous current is the vector sum of the three branch currents.

$$i_t = i_R + i_L + i_C.$$

Now $i_R = v/R = 100 \sin(5000\,t + \pi/4)/25$

$= 4 \sin(5000\,t + \pi/4)$

$$i_L = \frac{1}{L}\int v\,dt = \frac{10^3}{2}\int 100 \sin\left(5000t + \frac{\pi}{4}\right) dt$$

$$= \frac{10^3 \times 100}{2}\left[\frac{-\cos(5000\,t + \pi/4)}{5000}\right] = -10 \cos(5000\,t + \pi/4)$$

$$i_C = C\frac{dv}{dt} = C.\frac{d}{dt}[100 \sin(5000\,t + \pi/4)]$$

$= 30 \times 10^{-6} \times 100 \times 5000 \times \cos(5000\,t + \pi/4) = 15 \cos(5000\,t + \pi/4)$

$i_t = 4 \sin(5000\,t + \pi/4) - 10 \cos(5000\,t + \pi/4) + 15 \cos(5000t + \pi/4)$

$=$ **$4 \sin(5000\,t + \pi/4) + 5 \cos(5000\,t + \pi/4)$**